VCG Example
- User 1 needs both A and B and has valuation z.
- User 2 needs A and has valuation x
- User 3 needs B and has valuation y

If z > x+y
- 1 gets A and B and pays x+y
If z < x+y
- 2 gets A and 3 gets B
  - If z > y, 2 pays z-y
  - If y > z, 2 pays 0

Static vs. Multi Stage
- Static Games
  - Players choose strategies simultaneously, without knowing what the others do.
- Multi-Stage
  - Game is played in multiple rounds
  - Players may see how others played in previous rounds.
  - That information helps choose how to play in the next round.
  - A strategy is a full specification of what actions to take in each stage, as a function of the observations from previous stages.

Repeated Innovating Firms Game (Repeated Prisoners Dilemma)
Firm A
Firm B
- Recall, Both firms innovate in the one-shot game.
  - Combined reward is 0.
  - If they had both stagnated instead, their combined reward would have been 4.
- What happens if the game is repeated?
  - Same game is repeated every year forever.
  - NPV of future payoffs is discounted by a discount factor β.

Repeated Innovating Firms Game
- Strategy: I will stagnate as long as you do.
- Threat: If you choose to innovate once, I will innovate forever thereafter.
- This is a NE if β > 1/3

Repeated Innovating Firms Game (Repeated Prisoners Dilemma)
Firm A’s:
\[ \sum_{n=0}^{\infty} \beta^n A(x_n, y_n) \]
Firm B’s:
\[ \sum_{n=0}^{\infty} \beta^n B(x_n, y_n) \]
Where
- \( x_n \): Albert’s action in slot n
- \( A() \): Albert’s payoff function
- \( y_n \): Bob’s action in slot n
- \( B() \): Bob’s payoff function
- β: Discount factor

Repeated Innovating Firms Game
- Proof:
  - Suppose at time t, A innovates
    - B retaliates by innovating forever thereafter
    - A is forced to innovate at times t+1, t+2, ... as well
  - A’s net payoff
    - One Step Reward
    - Future Consequences
    \[ \Delta = \beta (3 - 2) + \sum_{n=t+1}^{\infty} \beta^n (0 - 2) \]
    \[ = \beta - 2 \frac{\beta^{t+1}}{1 - \beta} \]
    \[ = \frac{\beta^t}{1 - \beta} (1 - \beta - 2\beta) \leq 0 \text{ when } \beta \geq \frac{1}{3} \]
Repeated Innovating Firms Game

- **Intuition**
  - When $\beta$ is large, future consequences of breaking the collusion agreement outweigh short term gain.
  - When $\beta$ is small, short term gain is more important than long term consequences.

SPE

- Player 2 can “threaten” to choose $R$ in stage 2 to get Player 1 to pick $R$ is stage 1.
  - But in the subgame starting in slot 2, Player 2 is compelled to pick $L$.
  - Player 2’s threat is not credible.
  - $(R,R)$ is indeed a NE, but not SPE.
  - Only $(L,L)$ is a SPE.

Repeated Innovating Firms Game

- We said that the following strategy profile is a Nash Equilibrium:
  - **Strategy**: I will stagnate as long as you do.
  - **Threat**: If you choose to innovate once, I will innovate forever thereafter.

Is it a SPE?

- Yes.
- In the subgame after the first deviation, it is rational to Innovate forever thereafter if you expect your opponent to do the same.

Repeated Innovating Firms Game

- “Folk Theorem” - Cooperating can be rational if games are repeated forever.
- However, threat strategies can be used to enforce other outcomes.

Claim: Any Reward vector in the green region can be enforced by an SPE.

Repeated Innovating Firms Game

- Proof:
  - Consider $v$ in the green region: $v = \sum_{j=1}^{4} \lambda_j r_j$, $v > (0,0)$
  - Pick integers $N_j$ that satisfy $\lambda_j \approx N_j/N$ for $j = 1, \ldots, 4$
  - $N = N_1 + \cdots + N_4$
  - They agree to play
    - $(I,I)$ the first $N_1$ steps
    - $(S,S)$ the next $N_2$ steps etc.
  - When someone deviates from the schedule, the other retaliates by playing $I$ forever thereafter.

Finitely Repeated Innovating Firms Game

- Suppose they play the game only $N$ times.
  - Is it a SP Equilibrium to play $(S,S)$ in all turns?
  - Consider the $N$th stage of the game.
    - In the $N$th stage, the players don’t have to worry about how their action affects the future.
    - Thus at the $N$th stage both players Innovate.
      - (This is a dominant strategy for both players.)
  - At time $N-1$, the players know their actions can’t affect the future.
    - Thus $(I,I)$ is again the dominant strategy.
Finitely Repeated Innovating Firms Game

- By induction,
  - The players play (I,I) in every slot.
  - Such a strategy profile is the only Sub-Game Perfect Nash Equilibrium (SPE).

Example: Cournot Competition

- Two firms produce goods
- Choose quantities \( q_1, q_2 \)
- Market clearing price
  - \( A - q_1 - q_2 \)
- Cost of production is \( C \) per unit
  - \( U_1(q_1, q_2) = (A - q_1 - q_2)q_1 - Cq_1 \)
  - \( U_2(q_1, q_2) = (A - q_1 - q_2)q_2 - Cq_2 \)
- Firm 1 Best Response
  - \( q_1^* = \arg \max q_1 U_1(q_1, q_2) = \frac{B}{2} \)
- Cournot Competition

Repeated Cournot Competition

Q: Is it possible for two firms to reach an agreement to produce \( B/4 \) instead of \( B/3 \) each?
- Consider the strategy: I will produce \( B/4 \) as long as you do. If you deviate, I will produce \( B/3 \) forever thereafter.

A firm has two choices in each round:
- Cooperate: produce \( B/4 \) and make profit \( B^2/8 \)
- Cheat: produce \( 3B/8 \) and make profit \( 9B^2/64 \)

But in the subsequent rounds, cheating will cause
  - its competitor to produce \( B/3 \) as punishment
  - its own profit to drop back to \( B^2/9 \)

Revised Cournot Competition

Is there any incentive for a firm not to cheat?
- Let’s look at the accumulated payoffs:
  - If it cooperates:
    \[ S_C = (1 + \delta + \delta^2 + \delta^3 + \ldots) B^2/8 = B^2/8(1 - \delta) \]
  - If it cheats:
    \[ S_D = 9B^2/64 + (\delta + \delta^2 + \delta^3 + \ldots) B^2/9 \]
    \[ = (9/64 + \delta(1 - \delta)) B^2/9 \]
  - So it will not cheat if \( S_C > S_D \) This happens only if \( \delta > 9/17 \).

Conclusion
  - If future return is valuable enough to each player, then strategies exist for them to play socially efficient moves.
Final Remarks

- Game Theory important tool in conceptualizing strategic interactions
  - Between Competing Firms
  - Buyers and Sellers
  - Other Interacting Agents

First Price Auction

- Suppose 2 bidders.
  - One has valuation 1, the other 0.5
- Suppose both bidders know the valuations of the others - “full information”
- What should they bid?

First Price Auction (cont’d)

- Consider a bidder j bidding in this auction, needing to choose bid $b_j$
  - Suppose everyone else bids: $aV_i$ where $a$ is some fraction we want to solve for.
  - Chance j wins:
    $$\prod_{i \neq j} p(b_j > aV_i) = \left(\frac{b_j}{a} \right)^{N-1}$$
  - Expected Payoff:
    $$\left(\frac{b_j}{a} \right)^{N-1} (v_j - b_j)$$

First Price Auction (Cont’d)

- $b_j = \left(\frac{v_j}{N-1} \right) N$
- Differentiate w.r.t. $b_j$
  $$N-1 \left[ \frac{b_j}{a} \right]^{N-2} (v_j - b_j) + \left( \frac{b_j}{a} \right)^{N-1} = 0$$
  $$\frac{(N-1)v_j - Nb_j}{a} = 0$$
  $$b_j = \frac{N-1}{N} v_j$$

Second Price Auction

- Suppose one item for bid
- Users value the item at
  - $V_i$
- They bid
  - $b_i$
- The highest bidder wins and pays the second highest bid.
Second Price Auction

- The second price auction is “incentive compatible”
- No user has an incentive to not bid their true valuation
  - Suppose user $i$ unilaterally lowers his bid $b_i < V_i$
  - Might lose auction to someone with lower valuation
  - If $i$ wins, his payment is still $b_i$ so he gained nothing by underbidding.

Mechanism Design

- A game to allocate resources to users
  - Can be a collection of items
  - User valuations can be complicated functions of which items she gets and in what quantities
- Users make bids and then a Principal agent determines allocation and payments
- Possible goal: Design the mechanism so that resources are allocated in a way that maximizes social welfare – the sum of everyone’s utilities
- Another Possible Goal: Design the mechanism so that Auctioneer (the Principal) maximizes revenue

Revelation Principle

- Any mechanism can be implemented in a way so that users are truth revealing
- Why

Vickery Clark Groves

- Idea:
  - Consider user $i$
    - Consider welfare of other users under optimal allocation if $i$ were not present
    - Consider welfare of other users under optimal allocation if $i$ were present
  - Make user $i$ pay the difference.

VCG Example

- User 1 needs both A and B and has valuation $z$.
- User 2 needs A and has valuation $x$.
- User 3 needs B and has valuation $y$.

  - If $z > x+y$
    - 1 gets A and B and pays $x+y$
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    - 2 gets A and 3 gets B
      - If $y > z$, 2 pays $y-z$
      - If $y < z$, 2 pays 0
Facts about VCG

- Any efficient truth-telling mechanism that works for a set of type profiles is a VCG mechanism
- VCG is not budget balanced

Auction Revenue

- What does a 2nd price auction generate in the same situation?
  - What is the distribution of the 2nd highest among uniforms?
    - If highest is x, conditional mean of next highest is x(N-1)/N
    - Integrating across density of 1st highest:
      \[
      \int_0^{N-1} \frac{N-1}{N} x^{N-1} dx = \frac{N-1}{N+1}
      \]

Revenue Equivalence Theorem

- If 2 auctions are designed such that
  - A bidder of a given type has the same chance of winning
  - A bidder of the lowest type gets the same expected utility
- Then,
  - The expected revenue to the auctioneer is the same!

Reserve Price

- Consider 2nd price auction with reserve price R
  - With chance N R^{N-1} (1-R)^1 exactly one above reserve price
    - Revenue with reserve: R
    - Revenue without reserve: ([N-1]/N) R
  - With chance (1-R)^N everyone below reserve price
    - Revenue with reserve: 0
    - Revenue without reserve: ([N-1]/(N+1)) R

Myerson Auction

- In 1981 Myerson derived the form of the "optimal" auction for wide class of settings.