1. The Plantain Republic (PR) is a country consisting of 3 equal population districts. A presidential candidate who wins the popular vote in the majority of the districts wins the presidency. A polling analyst tells you that the candidate from the Mammoth party has chance $\mu$ of winning the popular vote in each district, while the Burro party has a $1-\mu$ chance in each district. We will explore how dependencies between districts affect the odds of a candidate winning the presidency. Let each $X_i$, $i \in \{1,...,3\}$ be a random variable taking on the value 1 if the Mammoth party wins district $i$, and 0 otherwise. Note that $E(X_i) = \mu$, and $\text{Var}(X_i) = \sigma^2_{X_i} = \mu(1-\mu)^2 + (1-\mu)(0-\mu)^2 = \mu(1-\mu)$.

(a) Suppose the outcomes of the 3 district races are independent. (The random variables $X_1$, $X_2$, $X_3$ are independent.) What is the chance that the Mammoth candidate wins the presidency? Express your answer in terms of $\mu$.

(b) Suppose that the outcomes of the 3 district races are completely dependent and that specifically $X_1 = X_2 = X_3$. (The outcomes of each of the district races always comes out the same.) What is the chance that the Mammoth candidate wins? Again, express your answer in terms of $\mu$.

(c) Suppose that the for each pair $X_i$, $X_j$ with $i \neq j$, the correlation coefficient is the same value $\rho_{X_i,X_j} = r$. The correlation coefficient $\rho_{X_i,X_j}$ is a measure of how much random variables tend to vary in the same way together. The expression for correlation coefficient is

$$\rho_{X_i,X_j} = \frac{E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)]}{\sigma_{X_i}\sigma_{X_j}}.$$ 

Also suppose that $P(X_1 = 1, X_2 = 1, X_3 = 1 | X_1 = X_2 = X_3 = 1) = a$. What is the chance that the Mammoth candidate wins the presidency? Express your answer in terms of $\mu$, $r$ and $a$. **Hint:** Define the notation $p_{ijk} = P(X_i = i, X_j = j, X_k = k)$. Show that $\rho_{X_1X_2} = \frac{E[X_1X_2] - \mu^2}{\mu(1-\mu)} = r$ and $E[X_1X_2] = p_{110} + p_{111}$. Write similar expressions for $\rho_{X_1X_3}$ and $\rho_{X_2X_3}$. Also show that

$$P(X_1 = 1, X_2 = 1, X_3 = 1 | X_1 = X_2 = X_3 = 1) = a = \frac{p_{111}}{p_{110} + p_{111}}.$$

(d) Does your answer for part (c) correspond to the answer for part (a) if $r = 0$ and $a = \mu$?

(e) Does your answer for part (c) correspond to the answer for part (b) if $r = 1$ and $a = 1$?

(f) Suppose $\mu = 0.1$, $r = 0.9$, $a = 0.8$. Which scenario above leads to the highest chance for the Mammoth candidate to win?

2. Suppose you are trapped in Las Vegas. You have $100, but you need $200 to buy an airline ticket to San Jose. You decide to play roulette. In each round, you can bet as much as the current amount of money you have (your “fortune”), and you have a chance of 18/38 of winning what you bet, and 20/38 of losing what you bet. If your fortune ever hits zero the game ends. You want to design a scheme that maximizes your chances of making it home. Obviously if your fortune ever hits $200, you stop – you will not keep gambling in hopes of getting enough for a first class ticket!

(a) Suppose you bet the full $100 in round 1. What’s your chance of making it home to San Jose?

(b) Suppose now you bet $50 a time.

i. Describe carefully what $\Omega$ (the outcome space) is for this experiment. How do you compute the probability of each $\omega \in \Omega$?

ii. One feature of this situation is that your chance of making it to San Jose after playing for a while only depends on your current fortune. Let $h(150)$ be the chance you make it to San Jose given that you currently have $150. Obviously, your chances are at least 18/38 since you could just win the next round. However, you could lose the next round and still have better “luck” later. Write an expression for $h(150)$ as a function of $h(100)$ – the chance you get to San Jose if your fortune is $100$.

iii. Similarly, write an expression for $h(100)$ in terms of $h(150)$ and $h(50)$ – the chance you get to San Jose if your fortune drops to $50$. Hint: if you have $100 in the next round you will either win, get to $150 after which your chance of making it to San Jose is $h(150)$ -or- go to $50 and have chance $h(50)$ of ultimately getting to San Jose.

iv. Finally write an expression for $h(50)$ in terms of $h(100)$ and $h(0) = 0$. 

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v. Solve your set of three equations for \( h(150), h(100), \) and \( h(50). \)

(c) Is it better to bet your starting money $50 at a time, or bet the whole $100 in one go?

3. Suppose one rolls a fair 6 sided die over and over again until they get a 1. What are \( \Omega, \mathcal{F}, \) and \( P? \) Let \( X \) be the sum of all the rolls made, including the final 1. What is \( E[X]? \)

4. Suppose that \( X \) and \( Y \) are independent and identically distributed random variables with density \( f_X(x) \) and \( f_Y(y). \) Let \( Z := X + Y. \) What is the density of \( Z? \)

Hint: The joint density of \((X,Y)\) is \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \) because \( X \) and \( Y \) are independent. We can evaluate the probability that \( Z \leq z \) by integrating the joint density of \((X,Y)\) on the region that makes \( Z \leq z. \) In particular, we can write:

\[
P(Z \leq z) = P(X \leq z - Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x)f_Y(y)dxdy.
\]

Differentiate this with respect to \( z \) to get the density of \( Z. \)