Assignment 1 TIM 207, Random Process Models in Engineering
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Due: April 14, 2015

1. Suppose you are trapped in Las Vegas. You have $100, but you need $200 to buy an airline ticket to San Jose. You decide to play roulette. In each round, you can bet as much as the current amount of money you have (your “fortune”), and you have a chance of 18/38 of winning what you bet, and 20/38 of losing what you bet. If your fortune ever hits zero the game ends. You want to design a scheme that maximizes your chances of making it home. Obviously if your fortune ever hits $200, you stop – you will not keep gambling in hopes of getting enough for a first class ticket!

(a) Suppose you bet the full $100 in round 1. What’s your chance of making it home to San Jose?

(b) Suppose now you bet $50 a time.
   i. Describe carefully what \( \Omega \) (the outcome space) is for this experiment. How do you compute the probability of each \( \omega \in \Omega \)?
   ii. One feature of this situation is that your chance of making it to San Jose after playing for a while only depends on your current fortune. Let \( h(150) \) be the chance you make it to San Jose given that you currently have $150. Obviously, your chances are at least 18/38 since you could just win the next round. However, you could lose the next round and still have better “luck” later. Write an expression for \( h(150) \) as a function of \( h(100) \) – the chance you get to San Jose if your fortune is $100.
   iii. Similarly, write an expression for \( h(100) \) in terms of \( h(150) \) and \( h(50) \) – the chance you get to San Jose if your fortune drops to $50. Hint: if you have $100 in the next round you will either win, get to $150 after which your chance of making it to San Jose is \( h(150) \) -or- go to $50 and have chance \( h(50) \) of ultimately getting to San Jose.
   iv. Finally write an expression for \( h(50) \) in terms of \( h(100) \) and \( h(0) = 0 \).
   v. Solve your set of three equations for \( h(150), h(100), \) and \( h(50) \).

(c) Is it better to bet your starting money $50 at a time, or bet the whole $100 in one go?

Suppose one rolls a fair 6 sided die over and over again until they get a 1. What are \( \Omega, F, \) and \( P \)? Let \( X \) be the sum of all the rolls made, including the final 1. What is \( E[X] \)?

2. Suppose that \( X \) and \( Y \) are independent and identically distributed random variables with density \( f_X(x) \) and \( f_Y(y) \). Let \( Z := X + Y \). What is the density of \( Z \)?

Hint: The joint density of \( (X, Y) \) is \( f_{XY}(x, y) = f_X(x)f_Y(y) \) because \( X \) and \( Y \) are independent. We can evaluate the probability that \( Z \leq z \) by integrating the joint density of \( (X, Y) \) on the region that makes \( Z \leq z \). In particular, we can write:

\[
P(Z \leq z) = P(X \leq z - Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dxdy - \int_{-\infty}^{\infty} \int_{z-y}^{\infty} f_X(x)f_Y(y)\,dxdy.
\]

Differentiate this with respect to \( z \) to get the density of \( Z \) on the left and the desired quantity on the right.

3. Suppose that \( X \) is a random variable that takes values in the set \( \{1, 2, ..., N\} \). Show that

\[
E[X] = \sum_{n=1}^{N} P(X \geq n).
\]

Note: this fact is true even if \( X \) takes values in the set \( \{0, 1, 2, ...\} \). A related useful fact is that if \( X \) is a non-negative real valued random variable, i.e. it takes values on \([0, \infty)\), then \( E[X] = \int_{0}^{\infty} P(X > x)\,dx \).

Hint

\[
P(X \geq n) = \sum_{i=1}^{n} P(X = i).
\]

Substitute this into

\[
\sum_{n=1}^{N} P(X \geq n).
\]

How many time \( P(X = 1) \) appears in the sum? What about \( P(X = 2) \)? How does this compare to the formula for computing \( E[X] \)?