1. Suppose one rolls a fair 6 sided die over and over again until they get a 1. What are $\Omega$, $\mathcal{F}$, and $P$? Let $X$ be the sum of all the rolls made, including the final 1. What is $E[X]$?

2. Suppose that $X_1$ and $X_2$ are independent and identically distributed random variables with density $f_X(x)$. Let $Z := X + Y$. What is the density of $Z$?

3. Suppose that $X$ and $Y$ are independent uniformly distributed random variables on $[0,1]$ for $X$ and $[0,2]$ for $Y$. Let $Z := 2X + Y$. What is $E[X|Z]$? What is $E[X|Y]$?

4. Suppose that $X$ is a random variable that takes values in the set $\{1, 2, \ldots, N\}$. Show that

$$E[X] = \sum_{n=1}^{N} P(X \geq n).$$

Note: this fact is true even if $X$ takes values in the set $\{0, 1, 2, \ldots\}$. A related useful fact is that if $X$ is a non-negative real valued random variable, i.e. it takes values on $[0, \infty)$, then $E[X] = \int_{0}^{\infty} P(X > x)dx$.

5. (a) Let $X := \sum_{i=1}^{n} 1_{A_i}$ where each $A_i$ is an event, and $1_{A_i}$ is the indicator random variable that takes on the value 1 if $A_i$ happens and 0 otherwise. Derive a formula for the variance of $X$ in terms of $P(A_i)$ and $P(A_i \cap A_j)$, $i \neq j$.

(b) If $k$ balls are put at random into $n$ boxes, what is the variance of $X$, the number of boxes with (strictly) less than 2 balls in them?

6. (a) Let $\phi : \mathbb{R} \to [0, \infty)$ be an increasing function, let $x$ be any arbitrary real number and suppose $\phi(x) \neq 0$. Show that

$$P(X \geq x) \leq \frac{E[\phi(x)]}{\phi(x)}.

Hint: fix $x$, and define

$$h(y) = \begin{cases} 0 & y < x \\ \phi(x) & y \geq x. \end{cases}$$

Note that $h(y) \leq \phi(y)$ for all $y$. Use the integral definition of expected value to show that $E[h(X)] \leq E[\phi(X)]$. For this problem, you may assume that $X$ has a density $f_X(x)$, but the result you are proving is actually true even if $X$ does not have a density.

(b) Use the fact you proved in part (a) to show that if $X$ is a non-negative random variable with mean $E[X]$,

$$P(X \geq x) \leq \frac{E[X]}{x}.$$  

This is known as the Markov Inequality.

(c) Use the fact that you proved in part (a) to show if $X$ is any real valued random variable with mean $E[X]$ and variance $\text{var}(X)$,

$$P((X - E[X])^2 \geq x) \leq \frac{\text{var}(X)}{x}.$$  

This is known as the Chebyshev Inequality.

7. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a convex function, that is,

$$\alpha g(x) + (1 - \alpha)g(y) \geq g(\alpha x + (1 - \alpha)y)$$

for all $\alpha \in (0,1)$ and $x, y \in \mathbb{R}$. Also suppose that $X$ is a random variable with density $f_X(x)$. Show that

$$E[g(X)] \geq g(E[X]).$$

This useful fact is known as Jensen’s inequality.

Hint: Let $m = E[X]$. Observe that $\frac{1}{2}g(m + \epsilon) + \frac{1}{2}g(m - \epsilon) \geq g(m)$ and this in turn implies that $g(m) - g(m - \epsilon) \leq g(m + \epsilon) - g(m)$. Therefore

$$\lim_{\epsilon \downarrow 0} \frac{g(m) - g(m - \epsilon)}{\epsilon} \leq \lim_{\epsilon \downarrow 0} \frac{g(m + \epsilon) - g(m)}{\epsilon}.$$  

Pick $a$ to be any number between the two limits define the affine function $l : \mathbb{R} \to \mathbb{R}$ such that $l(x) = a(x - c) + g(c)$. Show that $g(x) \geq l(x)$ for all $x$. 

Assignment 1 TIM 207, Random Process Models in Engineering

Instructor: John Musacchio

Due: April 16, 2013