Homework 5
DUE November 8, 2016

Part I. Word Problems

1. For the symmetric matrix game below:

\[
\begin{array}{cccc}
 & A & B & C & D \\
A & 45 & 0 & 60 & 10 \\
B & 60 & 45 & 0 & 10 \\
C & 0 & 60 & 45 & 10 \\
D & 45 & 45 & 45 & 0 \\
\end{array}
\]

(a) Write down an equation (expressed in terms of population frequencies of each strategy — \(p_A, p_B, p_C, p_D\)) that equates the fitness (or expected payoff) of strategy A to strategy B. Then find another equal fitness equation for strategies B and C, then for C and D and finally for D and A.

\[
\begin{align*}
\text{EU}(A) &= 45p_A + 00p_B + 60p_C + 10p_D \\
\text{EU}(B) &= 60p_A + 45p_B + 00p_C + 10p_D \\
\text{EU}(C) &= 00p_A + 60p_B + 45p_C + 10p_D \\
\text{EU}(D) &= 45p_A + 45p_B + 45p_C + 00p_D
\end{align*}
\]

The system of equations:

\[
\begin{align*}
-15p_A - 45p_B + 60p_C &= 0 \quad \text{eq 1} \quad \text{EU}(A) - \text{EU}(B) \\
60p_A - 15p_B - 45p_C &= 0 \quad \text{eq 2} \quad \text{EU}(B) - \text{EU}(C) \\
-45p_A + 15p_B + 10p_D &= 0 \quad \text{eq 3} \quad \text{EU}(C) - \text{EU}(D) \\
+45p_B - 15p_C - 10p_D &= 0 \quad \text{eq 4} \quad \text{EU}(D) - \text{EU}(A) \\
p_A + p_B + p_C + p_D &= 1 \quad \text{eq 5} \quad \text{4-simplex}
\end{align*}
\]

(b) Use those equations to find all points in the 4-simplex for which all four strategies have equal fitness.

The system of equation solves at \(p_A = p_B = p_C = \frac{1}{6}\) and \(p_D = \frac{1}{2}\).

One easy way is to notice that A, B and C are symmetric, they must have same probability. Therefore, we directly go to \(p_A = p_B = p_C = p\) and \(p_D = 1 - 3p\). We will need only eq3 or eq4 to get to the result.
(c) Use your results in (b) to help find all NE and all ESS of the game.

\[
\text{NE} = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2} \right)
\]

The NE is not ESS. Counter example, \( p = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2} \right) \) let \( q = \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right) \), and we can see that \( F(p, p) < F(q, p) \).

2. Write up your solution to Harrington Chapter 16, Problem 7.

7. Consider the game shown here. Find all ESS’s.

\[
\begin{array}{ccc}
\text{Player 2} \\
& w & x & y \\
\hline
w & 3,3 & 2.5 & 2.2 \\
x & 5.2 & 1.1 & 0.0 \\
y & 2.2 & 0.0 & 1.1 \\
\end{array}
\]

**ANSWER:** First note that \( w \) strictly dominates \( y \). Hence, the set of ESS’s for this game is the same as that for the game when \( y \) is eliminated, as shown here.

\[
\begin{array}{cc}
\text{Player 2} \\
& w & x \\
\hline
w & 3,3 & 2.5 \\
x & 5.2 & 1.1 \\
\end{array}
\]

This game has no symmetric pure-strategy Nash equilibria, and a symmetric mixed-strategy Nash equilibrium is defined (where \( p \) is the probability of choosing \( w \)) by

\[
3 \times p + 2 \times (1 - p) = 5 \times p + 1 \times (1 - p) \Rightarrow p = \frac{1}{3}
\]

For \( p = \frac{1}{3} \) to be a (weak) ESS, it must satisfy for all \( q \neq p \),

\[
p \times [3 \times q + 2 \times (1 - q)] + (1 - p) \times [5 \times q + 1 \times (1 - q)] \\
> q \times [3 \times q + 2 \times (1 - q)] + (1 - q) \times [5 \times q + 1 \times (1 - q)].
\]

This can be rearranged to

\[
(p - q) \left( \frac{1}{3} - q \right)^3 > 0.
\]

Inserting \( p = \frac{1}{3} \), it becomes

\[
\left( \frac{1}{3} - q \right)^2 \left( \frac{1}{3} - q \right) > 0,
\]

which does indeed hold for all \( q \neq \frac{1}{3} \). This game then has a unique ESS, with \( w \) chosen with probability \( \frac{2}{3} \) and \( x \) chosen with probability \( \frac{1}{3} \).
3. Recall the pitcher-batter population game in Harrington Chapter 17, page 624. Write out an excel worksheet to do discrete replicator dynamics for this two population game. You need not use the messy formulas given in the following pages of Harrington. Just use the spreadsheet to calculate for periods \( t = 1, 2, \ldots, 200 \) the fitnesses of right- (R) and left- (L) handed batters and of right and left-handed pitchers, given the current (period \( t \)) shares \( p^t \) and \( (1 - p^t) \) of R and L pitchers and the shares \( b^t \) and \( (1 - b^t) \) of R and L batters. Then use the replicator equation to update the shares \( p^{t+1}, p^{t+1}, \text{ etc.} \)

See excel sheet on the website.

4. Write up your solution to Harrington Chapter 17, Problem 6.

6. Return again to the 10-period Repeated Prisoners' Dilemma in Section 17.5. Now suppose there are these three strategies: (1) Defector; (2) Tit for Tat; and (3) Sneaky Tit for Tat. The first two strategies are as previously defined, while Sneaky Tit for Tat is exactly like Tit for Tat, except that it always chooses defect in the 10th period, regardless of what its partner chose in period 9. Let \( p^t \) and \( p^t \) denote the fraction of the population in generation \( t \) that are endowed with Sneaky Tit for Tat and Tit for Tat, respectively.

   a. Derive the fitness matrix. (This is the analogue to Figure 17.13.)

   \[ \begin{array}{c|ccc}
       & \text{Defector} & \text{Sneaky Tit for Tat} & \text{Tit for Tat} \\
       \hline
       \text{Defector} & 30,30 & 35,28 & 35,28 \\
       \text{Sneaky Tit for Tat} & 28,35 & 48,48 & 53,46 \\
       \text{Tit for Tat} & 28,35 & 46,53 & 50,50 \\
   \end{array} \]

   b. Compare the fitness of each pair of strategies.

   **ANSWER:** Average fitness of Tit for Tat exceeds that of Defector if and only if

   \[
   48p^t_S + 53p^t_T + 28(1 - p^t_S - p^t_T) > 35p^t_S + 35p^t_T + 30(1 - p^t_S - p^t_T) \\
   13p^t_S + 17p^t_T > 2 \iff p^t_S > \frac{2 - 17p^t_T}{13}.
   \]

   Average fitness of Sneaky Tit for Tat exceeds that of Tit for Tat if and only if

   \[
   48p^t_S + 53p^t_T + 28(1 - p^t_S - p^t_T) > 46p^t_S + 50p^t_T + 28(1 - p^t_S - p^t_T) \\
   2p^t_S + 3p^t_T > 0.
   \]

   Thus, as long as not all are Defectors, then Sneaky Tit for Tat has higher fitness than Tit for Tat. Average fitness of Sneaky Tit for Tat exceeds that of Defector if and only if

   \[
   48p^t_S + 53p^t_T + 28(1 - p^t_S - p^t_T) > 35p^t_S + 35p^t_T + 30(1 - p^t_S - p^t_T) \\
   13p^t_S + 20p^t_T > 2 \iff p^t_S > \frac{2 - 20p^t_T}{15}.
   \]

   c. Is a population mix in which all are endowed with Tit for Tat an attractor?

   **ANSWER:** No, for as long as the population is not all Defectors, then the fitness of Sneaky Tit for Tat exceeds the fitness of Tit for Tat. Hence, if \( p^t_T \) is close to 1, then the fraction of Sneaky Tit for Tat will grow faster than the fraction of Tit for Tat, which means the population mix will not return to having all Tit for Tat.
d. Is a population mix in which all are endowed with Defector an attractor?

**ANSWER:** Yes, for suppose $p_S + p_T = 0$, then

$$p_S < \frac{2 - 17p_T}{13} = \frac{2}{13}$$

so fitness of Defector exceeds that of Tit for Tat, and

$$p_S < \frac{2 - 20p_T}{15} = \frac{2}{15}$$

so fitness of Defector exceeds that of Sneaky Tit for Tat.

Hence, Defectors grow. Since almost all are Defectors, then average population fitness is approximately the average fitness of Defector, which means that, from the previous equations, the fitness of Sneaky Tit for Tat and Tit for Tat are both less than the average population fitness and thus are declining.

e. Is a population mix in which all are endowed with Sneaky Tit for Tat an attractor?

**ANSWER:** Yes, for suppose $p_S = 1$, then $2p_S + 3p_T = 2 > 0$

so fitness of Sneaky Tit for Tat exceeds that of Tit for Tat and

$$p_S > \frac{2 - 20p_T}{15} = \frac{2}{15}$$

so fitness of Sneaky Tit for Tat exceeds that of Defector.

Since the fitness of Sneaky Tit for Tat always exceeds that of Tit for Tat and Defector, then the fitness of Sneaky Tit for Tat is highest and so it is growing. As there are almost all Sneaky Tit for Tats, then average population fitness is approximately the average fitness of Sneaky Tit for Tat, which means that, from the previous equations, the fitness of Defector and Tit for Tat are both less than the average population fitness and thus are declining.