PART I Problems

1. Solution, Harrington chapter 5, problem 3, page 171

(a) If driver $i$ takes back roads and the total number of drivers taking the toll road is $t$ (so there are $100 - t$ drivers on the back roads), then driver $i$'s payoff is $1000 - 2(100 - t)$.

If driver $i$ takes the toll road and the total number of drivers on the toll road is $t$, then driver $i$'s payoff is $990 - t$, which nets out the cost of the toll.

(b) Consider a strategy profile in which the number of drivers on the toll road is $t$.

A driver who is taking the back roads (that is, she is one of those $100 - t$ drivers on the back roads) is acting optimally if and only if

$$1000 - 2(100 - t) \geq 990 - (t + 1).$$

$$\implies 3t \geq 189$$

$$\implies t \geq 63$$

A driver who is taking the toll road is acting optimally if and only if

$$990 - t \geq 1000 - 2(101 - t).$$

$$\implies 202 - 10 \geq 3t$$

$$\implies 64 \geq t$$

These two conditions imply that $64 \geq t \geq 63$.

Hence, there are two Nash equilibria: 36 drivers take the back roads and 64 take the toll road; or 37 drivers take the back roads and 63 take the toll road.

2. Minimum effort game. The payoff for player $i$ is:

$$2 \times \min_{j=1,\ldots,5}(S_j) - S_i$$

Let’s start by assuming all of the players have agreed to apply a maximum level of effort 7. Also that the minimum level they can play is 1.

If they can trust all the other players, then, at the time the game starts the player $i$ is faced with the choice as shown in table 1:

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<thead>
<tr>
<th>Effort</th>
<th>Payoff</th>
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<tbody>
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<td>1</td>
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<td>6</td>
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<td>7</td>
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Table 1: If everyone applies highest effort

But if there a chance that one of the person in this group will fail to keep their side of agreement and will play 6, even if they try hard, then the payoff for player $i$ are as shown in table 2. That is
Table 2: If even one person falters and plays 6

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<th>Effort</th>
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they would want to work only as hard as 6. Since this incentive exists for all the players, they will all want apply exactly the same level of effort, 6.

Again even if one person is likely to not be able to keep up with this agreement and will apply lower effort, all will want to go down to that level. And in a few rounds you can see how this will lead to everyone applying the least level of effort possible (1) and receiving least payoff (1).

All profiles with equal efforts are NE. If everyone has same abilities and can trust everyone (7,7,7,7,7). It could also be (1,1,1,1,1) in the case there is a belief that at least one of the members will not be able to keep up at any level above 1; or it could be any profile with equal efforts, in-between those extremes, depending on the beliefs of the players.


The husband’s best-reply function is to play $g_H$ such that they maximize their pay-off function assuming $g_W$ as given:

$$V_H(g_H, g_W) = 50g_H + \frac{1}{4}g_Hg_W - \frac{1}{2}g_H^2;$$

$$\Rightarrow \frac{\partial V_H(g_H, g_W)}{\partial g_H} = 50 + \frac{1}{4}g_W - g_H = 0;$$

$$\Rightarrow BR_H(g_W) = g_H^* = \frac{1}{4}g_W + 50. \text{ eq}(3.1)$$

Wife’s best-reply function is to play $g_W$ such that they maximize their pay-off function assuming $g_H$ as given:

$$V_W(g_H, g_W) = 50g_W + 2g_Hg_W - \frac{1}{2}g_W^2;$$

$$\Rightarrow \frac{\partial V_W(g_H, g_W)}{\partial g_W} = 50 + 2g_H - g_W = 0;$$

$$\Rightarrow BR_W(g_H) = g_W^* = 50 + 2g_H. \text{ eq}(3.2)$$

We can find Nash equilibrium by solving the best responses of both husband and wife (eq(3.1) and eq(3.2)) together. At Nash equilibrium,

$$50 + \frac{1}{4}g_W^* = \frac{1}{2}g_W^* - 25;$$

$$\Rightarrow g_W^* = 300.$$ 

Eq(3.2) becomes: $g_H^* = 125.$

Nash Equilibrium is at $g_H = 125$ and $g_W = 300.$
4. Alice and Bob’s venture.

(a) The payoff function for Alice \( V_A = sx_a^{1/3}x_b^{1/3} - 2x_a \).

Given \( x_b \) Alice’s BR will be \( x_a^* \) such that:

\[
\frac{\partial V_A}{\partial x_a} = \frac{1}{3}x_a^{-2/3}x_b^{1/3} - 2 = 0;
\]

\[
\implies x_b = \frac{6s}{x_a^{2/3}}; \\
\implies \text{BR}_a(x_b) = x_a^* = \frac{s^{3/2}}{6^{3/2}x_b^{1/2}}. \text{eq}(4.1)
\]

The payoff function for Bob \( V_b = (1 - s)x_a^{1/3}x_b^{1/3} - x_b \).

Given \( x_a \) Bob’s BR will be \( x_b^* \) such that:

\[
\frac{\partial V_b}{\partial x_b} = \frac{1}{3}x_a^{1/3}x_b^{-2/3} - 1 = 0;
\]

\[
\implies x_b^{-2/3} = \frac{3}{1 - s}x_a^{-1/3}; \\
\implies \text{BR}_b(x_a) = x_b^* = \frac{(1 - s)^{3/2}}{3^{3/2}}x_a^{1/2}. \text{eq}(4.2)
\]

(b) BR of Alice to Bob’s effort being 0, \( x_b = 0 \) is \( x_a = 0 \).

BR of Bob to \( x_a = 0 \) is \( x_b = 0 \).

Thus \((0,0)\) is a Nash Equilibrium.

(c) For a non-trivial Nash Equilibrium, we can solve eq(4.1) and eq(4.2) together:

\[
\frac{6^3}{s^3}x_a^2 = \frac{(1 - s)^{3/2}}{3^{3/2}}x_a^{1/2}.
\]

Square both sides and dividing by \( x_a \) (if \( x_a \neq 0 \)):

\[
\frac{6^6}{s^6}x_a^3 = \frac{(1 - s)^3}{3^3}; \\
\implies x_a^* = \frac{s^3(1 - s)}{108}; \\
\text{Using eq(4.2) } x_b^* = \frac{s(1 - s)^2}{54}.
\]

Thus \( \left( \frac{s^2(1 - s)}{108}, \frac{s(1 - s)^2}{54} \right) \) is a Nash Equilibrium.

(d) Combined payoff = \( x_a^{1/3}x_b^{1/3} - 2x_a - x_b = \frac{s(1 - s)}{27} \).

(e) Value of the venture = \( x_a^{1/3}x_b^{1/3} = \frac{s(1 - s)}{18} \).

Which can be maximized by maximizing \( s(1 - s) \implies s = 0.5 \).