MIDTERM EXAM
Answer Key

Instructions: In class, closed book, 95 minutes. Partial credit will be granted for brief, relevant remarks and for partial results, but not for core dumps. Points as marked; total is 50.

1. Consider the two player game described by the following payoff matrix.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1.3</td>
<td>4.4</td>
<td>2.2</td>
<td>6.1</td>
</tr>
<tr>
<td>M</td>
<td>0.4</td>
<td>3.2</td>
<td>0.0</td>
<td>5.5</td>
</tr>
<tr>
<td>D</td>
<td>1.2</td>
<td>5.3</td>
<td>2.2</td>
<td>1.6</td>
</tr>
</tbody>
</table>

(a) Does either player have a dominant strategy? [1pt]
No.

(b) Does either player have a dominated strategy? [1pt]
Yes. [0.5 pts]
c and M [0.5 pts]

(c) Which profiles survive iterated deletion of strictly dominated strategies? [3pts]
(U,b), (U,d), (D,b) and (D,d) survive the IDSDS.

(d) Find all Nash equilibria (NE) of the game. [5pts]
No pure strategy NE, as from finding BRs. [1 pt]

To find NE in mix strategies, equate expected payoffs and solve. [2 pt for mentioning how to do it]

$$E_u(U) = E_u(D)$$
$$5q^* + 1(1 - q^*) = 4q^* + 6(1 - q^*)$$
$$\Rightarrow 6q^* = 5$$
$$\Rightarrow q^* = \frac{5}{6}$$

Similarly
$$E_u(U) = E_u(D)$$
$$4p + 3(1 - p^*) = p + 6(1 - p^*)$$
$$\Rightarrow 6p^* = 3$$
$$\Rightarrow p^* = \frac{1}{2}$$

So unique NE in mixed strategies is \((p^*, q^*) = \left(\frac{1}{2}, \frac{5}{6}\right)\) [2 pts for reaching final solution]

2. Consider the 3-player game in extensive form above.
(a) Write out the strategy set of each player. [3pts]
\[S_1 = \{a, b\} \text{ [1 pt]}\]
\[S_2 = \{xx, xy, yx, yy\} \text{ [1 pt]}\]
\[S_3 = \{ccc, ccd, cdc, cdd, dcc, dcd, ddc, ddd\} \text{ [1 pt]}\]

(b) Find all subgame perfect NE for the game. [5pts]
0.5 pt each for BI of each of the 6 sub-games.

By backward induction:
Player 3 will play dcc (where d is the move at the subgame on left, c is the move at the information set, and c at the sub-game on right).
Knowing this, player 2 will play yx.
and with the above moves in mind player 1 will play a.

Unique SGPNE is \((a, yx, dcc)\) [2 pt for reaching here]
3. You are captain of a canoe. Entering a narrow channel from the opposite direction is an oil tanker ship. Both you (player C) and the oil tanker skipper (player T) have to decide whether to turn (L)eft or (R)ight. If one ship turns right and the other left, there will be a collision, in which case the canoe is destroyed, giving you a payoff of -5 and the oil tanker is undamaged, but the tanker skipper has to fill out an annoying accident investigation report form, giving her a payoff of -1. An (L, L) outcome is best for you since it positions you a little closer to the dock on the other end of the channel; the payoffs here are 2 for you and 1 for T. An (R,R) outcome helps T make an upcoming turn so here the payoff is 1 for C and 2 for T.

(a) The oil tanker is bigger and heavier, therefore once the skipper makes a decision between L and R she cannot change that decision later. Once the oil tanker decides, you (player C) sees how the oil tanker moved, and decide which way to go. Write out the extensive form of this game. [3pts]

(b) Continuing from part a, what are the strategy sets of each player? [3pts]

\[ S_T = \{L, R\} \] [1 pts]
\[ S_C = \{LL, LR, RL, RR\} \] [2 pts]

(c) Again continuing from part a, find the subgame perfect (or backwards induction) solution to the game. [4pts]

1 point each for 3 sub games. And 1 for reaching final solution.

(d) Let’s modify the situation slightly. Suppose now that before the oil tanker moves, you have an opportunity to radio the oil tanker skipper and tell her which way you intend to move. If you told her that you intend “to move left no matter what,” should the oil tanker skipper believe you? Describe why or why not. Can you think of anything creative that you could tell the oil tanker skipper to make it more believable that you really will go left? [2pts]

The oil tanker should not believe the threat. If the tanker moves right, it is in the best interest of the canoe to move right in the ensuing subgame. It is not in the best interest of the canoe to honor their threat to move left. [1 pt]

The canoe driver could tell the oil tanker something that indicates that the canoe won’t have any other choice but to turn left at their decision node – a commitment device. In particular, he could say that the rudder is jammed and only turns left. [1 pt]
(e) We modify the situation one last time. Now suppose that you (player C) are also operating a large boat. Therefore both you and player (T) have poor maneuverability and thus must SIMULTANEOUSLY choose between L and R. Assume that the payoffs for the various outcomes – (L,L),(L,R), (R,L), and (R,R) – are the same as before. What are the extensive form and strategic forms of this game? What are the strategy sets of each player? [3pts]

**Extensive form.** [1 pt]

Strategy sets: \( S_T = \{L, R\} \), \( S_C = \{L, R\} \) [1 pt]

**Strategic form.** [1 pt]

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & p & 1-p \\
q & L & 1.2 \\
 & R & -1.5 \\
1-q & L & -5.1 \\
 & R & 2.1 \\
\end{array}
\]

(f) For the game as described in part (e), find all pure strategy Nash equilibria, if any, as well as all mixed strategy Nash equilibrium, if any. [5pts]

**Pure strategy equilibria (by best response):** (L,L) and (R,R). [1 pt each]  

For mixed strategy equilibrium:

To find mixed first assume canoe plays \( l \) with prob. \( p \) and \( r \) with prob. \( 1-p \). Tanker plays \( L \) with prob. \( q \) and \( R \) with prob. \( 1-q \). [1 pt for stating the steps]

Then make tanker indifferent:

\[
E[U(L)|p] = 2p - 1 \\
E[U(R)|p] = 2 - 3p
\]

So we get \( p^* = 3/5 \) such that \( E[U(L)|p] = E[U(R)|P] \) [1 pt]

Finally make canoe indifferent:

\[
E[U(l)|q] = 2q - 5(1 - q) = 7q - 5 \\
E[U(r)|q] = 1 - 11q = -5q + 1(1 - q) = -6q + 1
\]

So we get \( q^* = 6/13 \), such that \( E[U(L)|q] = E[U(R)|q] \). [1 pt]

These two probabilities represent the mixed NE \((p, q)\).
4. It’s 2017 and the next “great recession” has started. Just as in 2008, the car industry is in serious trouble. As car czar of the United States, you are responsible for recommending a reorganization of the industry to reduce the losses the carmakers are suffering. Your demand forecaster tells you that you can think of all cars as being identical, and that the market price $p$ for cars depends on the total quantity $Q$ produced in the following way:

$$p = (40 - Q).$$

Suppose there are $n$ car companies, and each company chooses a quantity $q_i$ of cars to produce. It costs each car company 20 to make each car, and each car company has fixed “legacy” costs of 25 so that the payoff function for each company is:

$$V_i = (p - 20)q_i - 25.$$

(Note on units: The prices are in units of thousands of dollars, quantities in units of millions of cars, and payoffs in units of billions of dollars. The formulas above are consistent with this, so you can work with the formulas as given without adding lots of zeros to the end of all the numbers!)

(a) What is the best response function of company $i$? Express your answer in terms of $q_{-i}$, the sum of the quantities the companies other than $i$ produce. \(4 \text{pts}\)

Player $i$’s payoff has the form:

$$V_i = (20 - q_i - q_{-i})q_i - 25$$

This is quadratic in $q_i$, and hill shaped, so the best response is found by setting the derivative with respect to $q_i$ to 0. \(1 \text{ pt}\)

The value $q_i^\ast$ is the best response $BR_i$. \(1 \text{ pt}\)

$$\frac{dV_i}{dq_i} = 20 - q_{-i} - 2q_i$$

$$\frac{dV_i(q_i^\ast)}{dq_i} = 0$$

$$\Rightarrow 20 - q_{-i} - 2q_i^\ast = 0$$

$$\Rightarrow BR_i = q_i^\ast = 10 - \frac{1}{2}q_{-i} \quad \text{[2 pts]}$$

(b) Find a symmetric NE. \(4 \text{pts}\) In a symmetric NE it must be that all players play best responses to each other, and that everybody produces the same quantity. \(1 \text{ pt}\)

This requires that $q_{-i} = (n - 1)q_i^\ast$. Since there are $n$ players, and thus every player faces $n - 1$ opponents. \(1 \text{ pt}\)

Substituting this into the best response relation, we have:

$$BR_i = q_i^\ast = 10 - \frac{1}{2}(n - 1)q_i^\ast$$

$$\Rightarrow q_i^\ast = \frac{20}{n + 1} \quad \text{[2 pt]}$$

(c) What is the profit of each company in the equilibrium you find in part b? \(2\text{pts}\)

This is found by substituting the above value into the payoff function. \(1 \text{ pt}\)

$$V_i = (20 - nq_i^\ast)q_i^\ast - 25$$

$$= \frac{400}{(n + 1)^2} - 25. \quad \text{[1 pt]}$$
(d) The president wants as much competition as possible while having an industry that doesn’t lose money. What is the largest number of companies in the car business possible and still have each company not lose money (not have a negative payoff)? [2pts]

From part C, we see that the profit of each company decreases in $N$ in the symmetric NE, so the largest $N$ that makes the payoff non-negative should result in a zero payoff. [1 pt for reasoning]

(If this procedure gives us a non-integer $N$, we’d have to round down to find an integer that makes $V_i$ non-negative. However, it turns out we get an integer $N$ without rounding.) The algebra works as follows:

\[
V_i = \frac{400}{(n+1)^2} - 25 = 0
\]
\[
\Rightarrow \frac{400}{(n+1)^2} = 25
\]
\[
\Rightarrow \frac{20}{n+1} = \pm 5 \quad \text{[taking square root on both sides]}
\]
\[
\Rightarrow n + 1 = \pm 4
\]
\[
\Rightarrow n = 3, -5
\]

Negative value is meaningless in this context, thus $n = 3$. [1 pt for algebra]