Problem 6, Harrington Chapter 10

a. Derive the conditions whereby it is a symmetric BNE for a candidate to enter only when she has a low personal cost from doing so.

**ANSWER:** It is a symmetric Bayes-Nash equilibrium for a senator to enter only when she has low personal cost if and only if

\[
\begin{align*}
\text{Low type: } p(v_2 - f_L) + (1 - p)(v_1 - f_L) &\geq 0 \Rightarrow pv_2 + (1 - p)v_1 \geq f_L, \\
\text{High type: } 0 \geq p(v_2 - f_H) + (1 - p)(v_1 - f_H) \Rightarrow f_H \geq pv_2 + (1 - p)v_1.
\end{align*}
\]

These two conditions can be combined to yield

\[f_H \geq pv_2 + (1 - p)v_1 \geq f_L.\]

This condition holds when \( p \) is close to 1.

b. Derive the conditions whereby it is a symmetric BNE for a candidate to enter for sure when she has a low personal cost and to enter with some probability strictly between 0 and 1 when she has a high personal cost.

**ANSWER:** Consider a symmetric strategy profile such that (1) if the candidate has low cost, she enters and (2) if she has high cost, she enters with probability \( q \).

\[
\begin{align*}
\text{Low type: } [p + (1 - p)q] &\geq (1 - q)(v_1 - f_L) \\
\Rightarrow [p + (1 - p)q]v_2 + (1 - p)(1 - q)v_1 &\geq f_L. \\
\text{High type: } 0 \geq [p + (1 - p)q] &\geq (1 - q)(v_1 - f_H) \\
\Rightarrow q = \frac{pv_2 + (1 - p)v_1 - f_H}{(1 - p)(v_1 - v_2)} &\Leftrightarrow q = \frac{(1 - p)(v_1 - v_2) + v_2 - f_H}{(1 - p)(v_1 - v_2)}. 
\end{align*}
\]

[SOL10.6.1]

With the high type, she must be indifferent between entering and not entering. For this to be an equilibrium, the derived value for \( q \) must lie between 0 and 1:

\[
0 < \frac{(1 - p)(v_1 - v_2) + v_2 - f_H}{(1 - p)(v_1 - v_2)} < 1
\]

\Rightarrow \frac{(1 - p)(v_1 - v_2) + v_2 - f_H}{(1 - p)(v_1 - v_2)} > 0 \Rightarrow pv_2 + (1 - p)v_1 > f_H. 

[SOL10.6.3]

If we substitute (SOL10.6.2) into (SOL10.6.1), the expression becomes \( f_H(v_1 - v_2) \geq f_L(v_1 - v_2) \), which always holds true. This means (SOL10.6.2) implies (SOL10.6.1), so the only condition we need is (SOL10.6.2). If \( pv_2 + (1 - p)v_1 > f_H \), then there is an equilibrium in which a low-cost Senator enters and a high-cost one randomizes. If \( f_H \geq pv_2 + (1 - p)v_1 \geq f_L \), then, as we know from part (a), there is an equilibrium in which a low-cost Senator enters and a high-cost one does not.

c. Find some other BNE distinct from those described in (a) and (b).

**ANSWER:** There is also an asymmetric equilibrium in which Senator 1 always enters and Senator 2 enters only when she has low cost. Given Senator 2’s strategy, it is always optimal for Senator 1 to enter if and only if

\[
\begin{align*}
\text{Low type: } p(v_2 - f_L) + (1 - p)(v_1 - f_L) &\geq 0 \Rightarrow pv_2 + (1 - p)v_1 \geq f_L, \\
\text{High type: } p(v_2 - f_H) + (1 - p)(v_1 - f_H) &\geq 0 \Rightarrow pv_2 + (1 - p)v_1 \geq f_H.
\end{align*}
\]

Given Senator 1’s strategy, it is optimal for Senator 2 to enter only when she has low personal cost if and only if

\[
\begin{align*}
\text{Low type: } v_2 - f_L &\geq 0 \\
\text{High type: } v_2 - f_H &\leq 0.
\end{align*}
\]

The last two inequalities obviously hold. Hence, this equilibrium exists if \( pv_2 + (1 - p)v_1 \geq f_H \).
a. Find a separating PBNE.

**ANSWER:** Consider the following separating strategy profile where $A' > A$:

**Owner’s strategy:**
- If of high quality, then spend $A'$ on advertising.
- If of low quality, then spend $A$ on advertising.

**Customer’s strategy:**
- If advertising is at least $A'$, then go to the restaurant.
- If advertising is less than $A'$, then do not go to the restaurant.

**Customer’s beliefs:**
- If advertising is at least $A'$, then the restaurant is high quality with probability 1.
- If advertising is less than $A'$, then the restaurant is low quality with probability 1.

Starting with the customer’s beliefs, consistency requires that the customer believe the restaurant is high quality when she observes advertising of $A'$ and is low quality when she observes advertising of $A''$. Thus, these beliefs are consistent. Turning to her strategy, first note that the customer finds it optimal to go to the restaurant if she believes it is high quality—realizing a payoff of 70 versus 0 from not going—and optimal not to go if she believes it is low quality (realizing a payoff of −20 versus 0). According to her beliefs, the restaurant is high quality if advertising is at least $A'$. As her strategy has her go to the restaurant if advertising is at least $A'$, prescribed behavior is optimal. She believes it is of low quality when advertising falls below $A'$; again her strategy is optimal.

Finally, consider the owner’s strategy. If the restaurant is of high quality, the payoff from advertising $A$ is $30 - A$ when $A \geq A'$ and zero when $A < A'$. Hence, advertising $A'$ is optimal if and only if $A' < 30$. When the restaurant is of low quality, the payoff from advertising $A$ is $15 - A$ when $A \geq A'$ and zero when $A < A'$. Hence, advertising $A''$, when $A'' < A'$, is optimal if and only if $A'' = 0$ and $A' \geq 15$. That is, if $A' < 15$, then the restaurant earns a higher payoff (of $15 - A'$) by advertising $A'$ than by advertising $A''$ (which results in a payoff of $-A''$). And if advertising $A'$ means no customers are going to come—so the payoff is $-A''$—then optimality requires $A'' = 0$. There is no point in advertising if it doesn’t deliver any customers. In sum, this is a separating perfect Bayes-Nash equilibrium if and only if $A'' = 0$ and $15 \leq A' \leq 30$. Note that advertising provides no direct information. Rather it is a signal of a restaurant’s quality. A high-quality restaurant is willing to advertise more to induce a customer to try its food because it knows the customer will return in the future.

b. At a separating PBNE, what is the maximum amount of advertising that a restaurant conducts? What is the minimum amount?

**ANSWER:** With this separating equilibrium, advertising cannot exceed 30 and cannot be less than 15. If it exceeds 30, then a high-quality restaurant would prefer not to advertise at all. If it is less than 15, then the low-quality restaurant would imitate the high-quality restaurant.
c. Find a pooling PBNE.

**ANSWER:** Consider the following separating strategy profile.

*Owner's strategy:*

If restaurant is of high quality, then spend \( A^o \) on advertising.
If restaurant is of low quality, then spend \( A^o \) on advertising.

*Customer's strategy:*

If advertising is at least \( A^o \), then go to the restaurant.
If advertising is less than \( A^o \), then do not go to the restaurant.

*Customer's beliefs:*

If advertising is at least \( A^o \), then the restaurant is high quality with probability \( \frac{1}{2} \).
If advertising is less than \( A^o \), then the restaurant is high quality with probability \( p \).

Since the customer's beliefs are the same as her prior beliefs when \( A = A^o \), beliefs are consistent. Given those beliefs, it is optimal to go to the restaurant when \( A \geq A^o \) if and only if:

\[
\frac{1}{2} \times 70 + \frac{1}{2} \times (-20) \geq 0,
\]

where the left-hand expression is the expected payoff from going, which equals 25, and the right-hand expression is the payoff from not going. This inequality holds. It is optimal not to go to the restaurant when \( A < A^o \) if and only if

\[
p \times 70 + (1 - p) \times (-20) \leq 0,
\]

\[
p \leq \frac{2}{9}.
\]

Thus, we must have \( p \leq \frac{2}{9} \). For the restaurant, it either wants to advertise \( A^o \) (the minimum amount necessary to induce a customer to come) or zero. The former yields a higher expected payoff when:

High-quality restaurant: \( 30 - A^o \geq 0 \) or \( A^o \leq 30 \)
Low-quality restaurant: \( 15 - A^o \geq 0 \) or \( A^o \leq 15 \).

Thus, we need \( A^o \leq 15 \).

d. At a pooling PBNE, what is the maximum amount of advertising?

**ANSWER:** The maximum amount of advertising at a pooling equilibrium is 15. If it exceeds 15, then the low-quality restaurant owner would prefer to advertise zero.