1. Attacker/Defender Game

   a) There is no pure strategy NE.

   b) Suppose that player 1 plays R1 with probability p, and player 2 plays C1 with probability q. Then

   \[ 3p - 2(1-p) = -4p + 3(1-p) \]
   \[ -3q + 4(1-q) = 2q - 3(1-q) \]

   which implies \( p^* = 5/12 \) and \( q^* = 7/12 \).

   Therefore, the mixed strategy NE is: player 1 plays R1 with probability 5/12 and R2 with probability 7/12; player 2 plays C1 with probability 7/12 and C2 with probability 5/12.

   c) The actual outcomes (on average, 61.2% C1, 49.7% R1) are close to the mixed strategy NE (58.3% C1, 41.7% R1).

2. Textbook problems

   7.5

   **ANSWER:** Note that \( c \) strictly dominates \( a \) and \( y \) strictly dominates \( z \). Thus, any Nash equilibrium in mixed strategies must assign zero probability to those dominated strategies. We can then eliminate them, so the reduced game is as shown in the figure below.

   ![Reduced Game](image)

   For this reduced game, \( b \) strictly dominates \( d \), so the latter can be deleted. The reduced game is as shown in the figure below.

   ![Further Reduced Game](image)

   This game has no pure-strategy Nash equilibria. To find the mixed-strategy Nash equilibria, let \( p \) denote the probability that player 1 chooses \( b \) and \( q \) denote the probability that player 2 chooses \( x \). The equilibrium conditions ensuring that players want to randomize are

   \[ q \times 5 + (1 - q) \times 2 = q \times 3 + (1 - q) \times 4 \Rightarrow q = \frac{1}{2} \]

   \[ p \times 1 + (1 - p) \times 7 = p \times 3 + (1 - p) \times 6 \Rightarrow p = \frac{1}{3} \]

   7.10
**ANSWER:** Let \( p \) denote the symmetric probability that a company enters. Given all other companies use this strategy, the expected payoff to a company for entering is

\[
(1 - p)^{n-1} \times (200 - 30) + \left[1 - (1 - p)^{n-1}\right] \times (40 - 30).
\]

\((1 - p)^{n-1}\) is the probability that all other companies do not enter, and 
\(1 - (1 - p)^{n-1}\) is the probability that at least one other company enters. This must be equated to that company's payoff from not entering, which is 60:

\[
(1 - p)^{n-1} \times (200 - 30) + \left[1 - (1 - p)^{n-1}\right] \times (40 - 30) = 60 \Rightarrow p = 1 - \left(\frac{5}{16}\right)^{\frac{1}{n-1}}.
\]

**7.18**

**ANSWER:** First note that Slap strictly dominates Kiss for player 2. Thus, any Nash equilibrium must have player 2 use pure strategy Slap. There are no dominated strategies for the other two players so the game looks like:

```
<table>
<thead>
<tr>
<th></th>
<th>Hug</th>
<th>Shove</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cuddle</td>
<td>1,2,4</td>
<td>3,4,1</td>
</tr>
<tr>
<td>Poke</td>
<td>5,3,2</td>
<td>0,5,4</td>
</tr>
</tbody>
</table>
```

Note that there is no pure-strategy Nash equilibrium. Let us then look to derive a Nash equilibrium in which players 1 and 3 randomize. Letting \( h \) denote the probability that player 3 hugs then player 1 is indifferent between her two pure strategies when

\[
\text{Cuddle: } h \times 1 + (1 - h) \times 3 = h \times 5 + (1 - h) \times 0 \Rightarrow h = \frac{3}{7}.
\]

Letting \( c \) denote the probability that player 1 cuddles then player 3 is indifferent between her two pure strategies when

\[
\text{Hug: } c \times 4 + (1 - c) \times 2 = c \times 1 + (1 - c) \times 4 \Rightarrow c = \frac{2}{5}.
\]

Thus, the unique mixed-strategy Nash equilibrium has player 1 cuddle with probability \( \frac{3}{7} \), player 2 slap for sure, and player 3 hug with probability \( \frac{3}{7} \).
a. Find all Nash equilibria. *(Hint: First derive the strategic form game.)*

**ANSWER:** The strategic form games are shown in the figure below.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& \text{Inquisitor: Torture} & \text{Galileo} & \text{Inquisitor: Do Not Torture} & \text{Galileo} \\
\hline
\text{Urban VIII} & \text{DNR} & 3,5,3 & 3,5,3 & 3,5,3 \\
& \text{R} & 5,3,4 & 5,3,4 & 4,1,5 & 1,2,1 \\
\hline
\end{array}
\]

The Nash equilibria are \((\text{DNR, DNC/DNC, torture}), (\text{R, C/C, torture}), (\text{R, C/DNC, torture}), (\text{DNR, DNC/C, do not torture}), (\text{DNR, DNC/DNC, do not torture})\).

b. Find all of the subgame perfect Nash equilibria.

**ANSWER:** In his last decision node (which is associated with the path refer → do not confess → torture), Galileo chooses do not confess. Given this choice, the Inquisitor chooses do not torture. At his first decision node (associated with Urban VIII having chosen refer), Galileo chooses do not confess. Finally, using the result just shown, Urban VIII chooses do not refer, as it produces payoff 3, which is greater than payoff 2 from playing refer. Hence, the unique subgame perfect Nash equilibrium is \((\text{DNR, DNC/DNC, do not torture})\).

c. For each Nash equilibrium that is not an SPNE, explain why it is not a SPNE.

**ANSWER:** There are four Nash equilibria that are not subgame perfect Nash equilibria. In Nash equilibria \((\text{DNR, DNC/DNC, torture})\) and \((\text{R, C/DNC, torture})\), the Inquisitor is making a nonoptimal decision by choosing to torture Galileo given Galileo plays do not confess in his last decision node. In Nash equilibria \((\text{R, C/C, torture})\) and \((\text{DNR, DNC/C, do not torture})\), Galileo is making a nonoptimal decision at his last decision node. He should play do not confess instead.
ANSWER: Consider the decision node faced by Guy when he kidnapped Orlando, Orlando told him the dirty secret, Guy released Orlando, and Orlando informed the police. If Guy chooses to tell the world about Orlando’s dirty secret, then Guy’s payoff is 2, and if he keeps quiet, then his payoff is 1; hence, he chooses tell. In other words, Guy takes pleasure in sticking it to Orlando for turning him in.

Now consider Orlando’s two final decision nodes. For the path kidnap → tell dirty secret → release, Orlando’s payoff is 1 from telling the police (as that action induces Guy to spill the beans about Orlando) and 2 from keeping quiet. Thus, Orlando chooses do not tell police. For the path kidnap → do not tell dirty secret → release, Orlando’s payoff is 4 from telling the police about Guy and only 2 from protecting Guy, so he chooses tell police. (In the latter situation, Guy can’t retaliate against Orlando, so Orlando has no incentive to keep quiet.) The game now looks like that in the figure below. Examining the final decision nodes, Guy should release Orlando when Orlando told Guy his dirty secret, and should kill him when he did not. Thus, the payoff to Orlando from baring his soul to Guy is 2, as it results in Guy’s releasing him, and his payoff from keeping quiet is 0, as then Guy kills him. Hence, Orlando chooses tell dirty secret.

Finally, at the initial decision node, Guy chooses kidnap, since it means a payoff of 5 (associated with Orlando telling Guy his dirty secret, Guy releasing Orlando, and Orlando keeping his mouth shut about Guy). The unique subgame perfect Nash equilibrium is (kidnap/release/kill/tell) for Guy and (tell dirty secret/do not tell police/tell police) for Orlando.
8.11

**ANSWER:** Once the foundry built the pair of custom iron gates, the value to the foundry if the grande dame did not buy them was nothing more than their value as scrap. Thus, the foundry would be willing to sell them for any price at least as great as their scrap value. The foundry failed to realize that once the gates were built, its bargaining position was seriously weakened. They should have written a contract that either required payment prior to the gates’ construction or that specified that if the grande dame did not make the payment of $1,400 upon delivery, then the foundry would destroy them. With that latter contract, after the gates were built, the grande dame’s choices would be to not have the gates or pay the $1,400 and have the gates. Presuming the latter is preferred, the foundry would receive payment. With either of these contracts, the negotiations are done upfront before the foundry builds the gates; at that point, the parties’ bargaining powers are comparable. Without a contract and once the gates are built, it is the grande dame who has much of the bargaining power.

9.2

**ANSWER:** In the subgame associated with Victor choosing *ask*, Hermione choosing *no*, and Ron choosing *ask*, Hermione will choose *yes*. In the subgame associated with Victor choosing *do not ask* and Ron choosing *ask*, Hermione will choose *yes*. Replacing the two final subgames with the Nash equilibrium payoffs, the situation is as shown in the following top figure.
The strategic form is shown in the following bottom figure.

<table>
<thead>
<tr>
<th>Victor</th>
<th>Hermione: Yes</th>
<th>Hermione: No</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ask</td>
<td>Do not ask</td>
</tr>
<tr>
<td></td>
<td>8*, 3, 6</td>
<td>7*, 6*, 5*</td>
</tr>
<tr>
<td></td>
<td>4*, 7*, 7*</td>
<td>5*, 4, 1*</td>
</tr>
</tbody>
</table>

Using the best-reply method, the subgame perfect Nash equilibria are (where the sequence is the strategy of Victor, Ron, and then Hermione) the following: (ask, do not ask, yes/yes/yes) and (do not ask, ask, no/yes/yes). Here the first action for Hermione occurs when Victor asks, the second action when Victor asks, she declines, and Ron asks her, and the third action occurs when Victor doesn’t ask and Ron asks her.

9.11

a. Find the SPNE. (Hint: The answer depends on the value of $n$).

**ANSWER:** When $n = 1$, there is a unique SPNE in which Norma Rae joins and the worker joins if and only if Norma Rae joins. By backward induction, if Norma Rae joins then 100% have joined if the worker joins and, therefore, the worker prefers to join. If Norma Rae does not join then only 50% have joined if the worker joins so the worker prefers not to join. Norma Rae then prefers to join.

When $n = 2$, consider the subgame in which Norma Rae joins. Regardless of what the other worker does, it is optimal for a worker to join as then at least two-thirds will have joined in which case the payoff is 3 while it is 2 from not joining. Hence, there is a unique Nash equilibrium for this subgame and it has both workers join. If Norma Rae does not join then there are two Nash equilibria: one in which both workers join and one in which neither worker joins. Working back to Norma Rae’s decision, if she joins then the two other workers subsequently join and her payoff is 3. If she does not join then her payoff is 2. Thus, she prefers to join. In sum, if $n = 2$ then there are two SPNE. One has Norma Rae join and the other two workers join if Norma Rae joins and do not join if Norma Rae does not join. The second one has Norma Rae join and the other two workers join whether or not Norma Rae joins. Both SPNE have all workers joining the union.

Now suppose $n \geq 3$. Whether or not Norma Rae joins, there are two Nash equilibria for the $n$-worker subgame: one in which everyone joins, and one in which no one joins. There are four SPNE: 1) Norma Rae joins and the other workers join if Norma Rae joins and do not join if Norma Rae does not join; 2) Norma Rae joins and the other workers join regardless of whether or not Norma Rae joins; 3) Norma Rae does not join and the other workers do not join if Norma Rae joins and join if Norma Rae does not join; and 4) Norma Rae does not join and the other workers do not join regardless of whether or not Norma Rae joins. The first two SPNE have everyone joining, the third SPNE has everyone but Norma Rae joining; and the fourth SPNE has no one joining.
b. Now suppose that payoffs are as follows. For Norma Rae, her payoff from not joining is 0, from joining when at least 2 people join is 1, and from joining when less than 2 people join is −1. For the other \( n \) workers, we will denote them as worker \#1, worker \#2, on through worker \#n. For worker \#i, his payoff from not joining is 0, from joining when at least \( 1 + i \) people join is 1, and from joining when less than \( 1 + i \) people join is −1, where \( i = 1, \ldots, n \). Find the SPNE.

**ANSWER:** The unique SPNE outcome is that everyone joins. Worker \#n joins only when his joining results in \( n + 1 \) union members which means the other \( n \) workers have to have joined (where the \( n \) workers comprise the other \( n - 1 \) workers plus Norma Rae). Worker \#n − 1 joins only if she thinks \( n \) people in total will join. If \( n - 1 \) people have joined when it comes to the decision of worker \#n − 1 then, by worker \#n − 1 joining, she brings the number to \( n \) so she joins. If only \( m \) (\(< n - 1\)) workers have joined then worker \#n − 1 will not join as doing so will only result in \( m + 1 \) (\(< n\)) workers joining because worker \#n will not join. By a similar logic, worker \#i will join if and only if \( i \) workers have joined when it comes to his turn. Worker \#i joins only if he thinks at least \( i + 1 \) people in total will join. If \( i \) workers have joined when it comes to the decision of worker \#i then his joining brings the number to \( i + 1 \) so he joins. If \( m \) (\(< i\)) workers have joined then, by worker \#i joining, the total will only end up amounting to \( m + 1 \) because workers \#i through \#n will not join. Thus, every worker will join if and only if all preceding workers have joined. Finally, Norma Rae joins because she knows if she joins, then worker 1 will join, and then worker 2 will join, and so on. Hence, SPNE results in everybody joining.