Assignment 3 ISM 207, Random Process Models in Engineering
Instructor: John Musacchio,
Due: February 5, 2008

1. (a) Suppose that \( Y \) is J.G. with zero mean and covariance
\[
K_Y = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}
\]
Find a matrix \( A \) such that \( W := A^{-1}Y \) satisfies \( W \sim N(0, I) \).

(b) What is the probability that \( 1.5y_1 + 2y_2 > 6 \)?

2. Suppose \( Y \) is J.G. with zero mean and covariance
\[
K_Y = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -3 \\ 0 & -3 & 6 \end{bmatrix}
\]
Find a matrix \( A \) and a JG random vector \( Z \) such that \( Y = AZ \) and \( \det(K_Z) \neq 0 \)

3. Let
\[
H(\omega) = \begin{cases} 
1 & \text{for } |\omega| \in [\omega_0 - \epsilon/2, \omega_0 + \epsilon/2] \\
0 & \text{otherwise}
\end{cases}
\]
Show that the inverse Fourier transform of \( H(\omega) \) is
\[
h(t) = 2 \cos(\omega_0 t) \frac{\sin(\epsilon t/2)}{\pi t}.
\]

4. (a) Suppose that \( W(t) \) is White Gaussian Noise with p.s.d. \( S_W(\omega) = N_0/2 \). Suppose that \( d(t) \) is a deterministic function satisfying \( \int_{-\infty}^{\infty} |d(t)|^2 dt = D \). Let
\[
Z = \int_{-\infty}^{\infty} d(t)W(t)dt.
\]
What is the distribution of \( Z \)?

(b) Suppose that your friend sends a 1 represented by the signal \( X(t) = s(t) \) with probability 0.5 or sends a 0 represented by the signal \( X(t) = 0 \) with probability 0.5. In either case, you receive the signal \( Y(t) = X(t) + W(t) \). Suppose you decide to build a detector. Your detector starts by computing the random variable \( Z := \int_{-\infty}^{\infty} d(t)Y(t)dt \). What is the distribution of \( Z \) conditioned on the event that your friend sends a 1? What is the distribution of \( Z \) conditioned on the event that your friend sends a 0?

(c) Write the MAP detector, based on the observation \( Z \). Use the results from your last homework assignment here.

(d) What is the probability of making an error if \( N_0/2 = 1 \) and \( D = 3 \)?

(e) Suppose now that you build your detector differently. Instead you compute \( Z' := \int_{-\infty}^{\infty} d'(t)Y(t)dt \), where \( d'(t) \) is a deterministic signal that is not equal to \( d(t) \) and satisfies
\[
\int_{-\infty}^{\infty} |d'(t)|^2 dt = D
\]
and
\[
\int_{-\infty}^{\infty} d'(t)d(t)dt = D/2.
\]
What is the distribution of \( Z' \) conditioned on the event that your friend sends a 1? What is the distribution of \( Z' \) conditioned on the event that your friend sends a 0?

(f) Write the MAP detector, based on the observation \( Z' \).

(g) What is the probability of making an error if \( N_0/2 = 1 \) and \( D = 3 \)?

(h) Which of the two detectors is better?