why probabilistic models?
- systems are unpredictable
- too complex to model deterministically

Objectives
- rational way to make decisions in face of uncertainty
- performance evaluation - average performance?

Applications
- estimation and detection, e.g., digital communication

\[
\begin{align*}
I \rightarrow & \text{Noise} \\
0 \rightarrow & \text{Signal}
\end{align*}
\]

Control

\[
\text{Control} \rightarrow \text{Plant} \rightarrow \text{Noise} \rightarrow \text{Output}
\]

Calls arrive

\[
\text{Telephone Network}
\]

Performance: blocking probability

\[
\text{1}
\]
A store sells WII.

Each day:
\[ P(\text{sell 2}) = 0.10 \]
\[ P(\text{sell 1}) = 0.30 \]
\[ P(\text{sell 0}) = 0.60 \]

Net revenue from each Wii sold: $10
Cost to hold each overnight: $2

Can order 0, 1, 2 units in evening and they arrive the next morning.

\[ X_i = \text{inventory at end of day } i \]

**Policy 1:** Order enough to have 2 units on hand every morning

\[ P(X_i = 0) = 0.10 \]
\[ P(X_i = 2) = 0.6 \]
\[ P(X_i = 1) = 0.3 \]

Expected revenue = (0.10) 20 + (0.30) 10 + (0.60) 0 = $5

Expected hold cost = (0.10) 0 + (0.30) 2 + (0.60) 4 = $3

\[ \underline{\text{Expected cost} = $12 \]}

**Policy 2:** Order 2 whenever inventory is 0 at end of the day.
we box a sale whenever we start a day with 20 units and have 2 customers try to buy 6-1.

\[
\left( \frac{3}{2} \right) \cdot (0.10) \cdot 10 = \frac{3}{2} = 0.428 \text{ lost revenue}
\]

\[
\text{expected revenue} = 5.5 - 0.428 = 4.572
\]

\[
\left( \frac{3}{2} \right) + (c) \cdot y = 15.5
\]

Hold cost:

\[
\left( \frac{3}{2} \right) \cdot 1 + \left( \frac{12}{35} \right) \cdot 2 \cdot 2 = 2.29 \text{ hold cost}
\]

\[
\text{expected profit} = 2.37
\]
A probability model has 3 components

\( \Omega \) = sample space, the set of all possible outcomes

\( \mathcal{F} \) = set of events, \( A \in \mathcal{F} \) is a subset of \( \Omega \)

\( p \) = probability measure assigning probability to each event

\( p : \mathcal{F} \rightarrow [0,1] \)

Axioms of \( \mathcal{F} \) (\( \mathcal{F} \) is a \( \sigma \)-field)

i) if \( A \in \mathcal{F} \), \( A^c \in \mathcal{F} \)

ii) if \( A \in \mathcal{F} \) for each \( i \), \( \bigcup_i A_i \in \mathcal{F} \)

Rules for \( p \)

i) \( p(A) \geq p(\emptyset) = 0 \quad \forall A \in \mathcal{F} \)

ii) \( p(\Omega) = 1 \)

iii) if \( A \in \mathcal{F} \), \( p(A^c) = 1 - p(A) \)

iv) if \( A_1, A_2, \ldots \) disjoint, \( p\left(\bigcup_i A_i\right) = \sum_i p(A_i) \)

Example: 2 coin flip

\( \Omega = \{HH, HT, TH, TT\} \)

\( \mathcal{F} = \) all subsets of \( \Omega = \{\{HH\}, \{HH, HT\}, \{H, H, TH, TT\}, \ldots\} \)

Valid probability measure:

\( \forall A \in \Omega \quad p(A) = \frac{1}{4} \)

if \( A = \{HT, TH\} \)

\( p(A) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \)

Example: Flip a coin until we get a tail

\( \Omega = \{T, HT, HHT, \ldots\} \)

\( \mathcal{F} = \) all subsets of \( \Omega \)
Let:

\[ p(\{H^R T\}) = \frac{1}{2} \]

Does this define a probability measure?

Yes (check axioms)

\[ A: \text{event that there are an even number of heads} \]

\[ p(A) = \sum_{i=0}^{\infty} \frac{1}{2} \left( \frac{1}{4} \right)^i = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{2}{3} \]
Conditional Probability

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Total Probability Theorem

Let \( B_1, B_2, \ldots \) be a countable partition of \( \Omega \)

\[
\begin{align*}
B_i \cap B_j &= \emptyset & i \neq j \\
\bigcup B_i &= \Omega
\end{align*}
\]

For any event \( A \)

\[ P(A) = \sum P(A \cap B_i) \cdot P(B_i) \]

Proof:

\[ P(C \cap A) = \sum P(C \cap A \cap B_i) = \sum P(C \mid A \cap B_i) \cdot P(A \cap B_i) \]

Bayes Rule

\[ P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \]

Proof:

\[ P(C \cap A) = P(C) \cdot P(A \mid C) = P(C) \cdot P(A) \]

Independence

Events \( A, B \) are independent if

\[ P(A \cap B) = P(A) \cdot P(B) \]

Random variable

A. r.v. is a mapping \( \Omega \to \mathbb{R} \)

E.g.: \( X(n) = \# \) heads in a sequence

For each \( x \in \mathbb{R} \), \( \{ n \in \mathbb{N} : X(n) \leq x \} \) is an event in \( \mathcal{F} \).