Example

\[
\begin{bmatrix}
1 & 0 \\
1/4 & 1/4 \\
1/4 & 3/4
\end{bmatrix}
\begin{bmatrix}
1/2 \\
1/4 \\
1/4
\end{bmatrix}^n
= \begin{bmatrix}
P(X_n = 1) \\
1^P(X_n = 2)
\end{bmatrix}_{X_0 = 1}
\]

\[
\begin{bmatrix}
0 & 1 \\
1/4 & 1/4 \\
1/4 & 3/4
\end{bmatrix}
\begin{bmatrix}
1/2 \\
1/4 \\
1/4
\end{bmatrix}^n
= \begin{bmatrix}
P(X_n = 1 | X_0 = 2) \\
P(X_n = 2 | X_0 = 2)
\end{bmatrix}_{X_0 = 2}
\]

\[
\begin{bmatrix}
P(X_0 = 1) & P(X_0 = 2)
\end{bmatrix}
\begin{bmatrix}
1/2 \\
1/4
\end{bmatrix}^n
= \begin{bmatrix}
P(X_n = 1) & P(X_n = 2)
\end{bmatrix}
\]

\[n = 2\]

\[
\begin{bmatrix}
3/8 & 5/8 \\
5/16 & 1/16
\end{bmatrix}
\]

\[n = 4\]

\[
\begin{bmatrix}
\frac{85}{256} & \frac{171}{256} \\
\frac{85}{256} & \frac{171}{256}
\end{bmatrix}
\]

\[n = 8\]

\[
\begin{bmatrix}
0.33334 & 0.66666 \\
0.33332 & 0.66667
\end{bmatrix}
\]

It appears that
\[
\lim_{n \to \infty} P(X_n = 1) = \frac{1}{3}
\]
\[
\lim_{n \to \infty} P(X_n = 2) = \frac{2}{3}
\]
for any initial condition.
Example 2

\[ 1 \rightarrow 2 \rightarrow 1 \]

This does not approach a steady state

\[ P(X_0 = 1) = 1 \]

\[ P(X_0 = 1) = 1 \text{ for never} \]

\[ P(X_0 = 1) = 0 \text{ for } n \text{ odd} \]

Definition: The period of state \( X \) is the largest number that will divide all \( n \) for which \( P^n(x, x) > 0 \)

\[ \left\langle \text{G.C.D. of } I_X \cdot n \geq 1 : P^n(x, x) > 0 \right\rangle \]

Lemma 4.1 If \( P(x, x) > 0 \), then \( X \) has period 1.

why \( 1 \in I_X \), the G.C.D must be 1

Lemma 4.2 If \( X \) has period 1, then no so that is no \( n \in I_X \)

For example

\[ I_X = \langle 3, 8, 11, 14 \rangle \]
Proof: Suppose \( k \leq k' \in \mathbb{N} \)

then \( I_k \) contains \( 2k, 2k+1, 2k+2 \)

\[ \vdots \]

\( I_k \) contains \( 3k, 3k+1, 3k+2, 3k+3 \)

\[ \vdots \]

\( I_k \) contains \( jk, jk+1, \ldots, jk+j \)

Pick \( j \geq k-1 \), so that these blocks overlap.

Thus from the point \( (k-1) \cdot k \), we have all consecutive integers in \( I_k \)

It remains to show that there exists 2 consecutive integers in \( I_k \)

Fact from number theory
If \( \text{GCD} \) of \( I_k \) is 1

then \( \exists c_1, \ldots, c_m \in \mathbb{N} \) and coefficients \( \in \mathbb{Z} \)

such that \( c_1 c_1 + \cdots + c_m c_m = 1 \)

Let \( a_i = c_i \cdot 1 (c_i > 0) \)
Let \( b_i = -c_i \cdot 1 (c_i > 0) \)

For example \( \langle 3, 8 \rangle \)

\[ (3) \times 3 + (-1) \cdot 8 = 1 \]

\[ 3 \times (3) = 8 \cdot (1) + 1 \]
thus we have two consecutive integers in $Tx$.
Lemma 4.3. If \( x \to y \) and \( y \to x \) then \( x, y \) have some period.

Proof: Suppose \( x \) has period \( d \leq C \).

\( y \) has period \( d \leq C \).

\( \exists k, m: p^k(x, y) > 0 \) and \( p^m(y, x) > 0 \).

Also \( p^{k+m}(x, x) \geq p^k(x, y) p^m(y, x) > 0 \).

Hence \( k+m \in I_x \), and thus \( k+m \) is a multiple of \( C \).

Let \( l \) be any integer with \( p^l(x, y) > 0 \) \( (l \notin I_y) \).

\( p^{k+l+m}(x, x) \geq p^k(x, y) p^l(y, y) p^m(y, x) > 0 \).

Thus \( k+l+m \in I_x \).

And \( k+l+m \) is a multiple of \( C \).

\( l \in I_y \) was arbitrary, hence \( C \) is a divisor of every element in \( I_y \).

But \( d \leq C \) is a divisor of \( I_y \).

\( C \) is possible.

Contradiction.
A stationary distribution $\pi$ is a vector satisfying the transition matrix $\pi^T P = \pi$ and $\sum_y \pi(y) = 1$.

Lemma 4.4: If $X_0$ has distribution $\pi$, then $X_n$ has distribution $\pi$ for all $n \geq 1$.

Proof: $P(X_n = y) = \pi^T P^n y$

$= \pi^T P^{n-1} y$

$= \pi^T P^{n-2} y$

$= \pi \cdot 1^n y$

If there exists a stationary distribution $\pi$ where all states $y$ that have $\pi(y) > 0$ are recurrent.

Proof: $N(y) = \# \text{visits to } y$
\[
\sum_x \pi(x) E_x N(y) = \sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x, y) = \sum_x \pi(x) E_x(\infty) = \sum_x p^n(x, y) = \sum_x p^n(x, y)
\]

Recall: \[E_x N(y) = \frac{p_{xy}}{1 - p_{xy}}\]

\[\sum_x \pi(x) \frac{p_{xy}}{1 - p_{xy}} = \infty\]

\[p_{xy} = 1 \implies y \text{ is recurrent}\]

A chain is **periodic** if all states have period 1.

A chain is **irreducible** if all states communicate with each other.
Theorem 4.1.5

Suppose \( \mathbf{P} \) is irreducible, aperiodic, and has stationary distribution \( \mathbf{\pi} \).

Then
\[
\lim_{n \to \infty} \mathbf{p}^n (x, y) \to \mathbf{\pi}(y)
\]