Renewal theory

\[ T_1, T_2, T_3, \ldots \text{ iid with dist FC } \]

\[ T_n = e_1 + \cdots + e_n \]

\[ N(t) = \min \{ n : T_n \leq t \} \]

\[ I = \{ X > 0 \} \]

\[ E_X = \int_0^\infty P(X > t) \, dt \]

\[ X = \int_0^\infty 1(X > t) \, dt \]

\[ E_X = \int_0^\infty P(X > t) \, dt \]

\[ E[N(t)] = \sum_{n=1}^{\infty} P(N(t) = n) = \sum_{n=1}^{\infty} P(T_n \leq t) \]

** Wald's Equation:** Let \( S_n = X_1 + X_2 + \cdots + X_n \) where \( X_1, X_2, \ldots, X_n \) i.i.d. \( N = EX_i \).

If \( N \) is a stopping time with \( EN < \infty \),

\[ E S_N = N \cdot EN \]

\[ S_N = \sum_{m=1}^{\infty} X_m 1(N \geq m) \]

the event \( N < m \) can be determined by time \( m-1 \).

Thus \( 1(N \geq m) \) is independent of \( X_m \).

\[ E S_N = \sum_{m=1}^{\infty} E X_m E 1(N \geq m) \]

\[ = N \sum_{m=1}^{\infty} P(N \geq m) = N \cdot EN \]

**Bounds on \( EN(t) \):**

\[ N = \min \{ n : T_n \leq t \} \text{ is a stopping time} \]

\[ = N_{(t)} + 1 \]

is \( N(t) \) a stopping time?

No!
\[ E \left( T_{\text{nu}} + 1 \right) = \nu E(N_{Ct} + 1) \]

\[ \epsilon < \nu(E N_{Ct} + 1) \]

\[ E N_{Ct} + 1 \geq \frac{\epsilon}{\nu} \]

Suppose \( \epsilon_i < m \)

\[ E \left[ N_{Ct} + 1 \right] \leq \frac{C^* + m}{\nu} \]
Strong law for renewal process: let $N = E\xi$ (mean inter-arrival time)

$$\frac{N(t)}{t} \to \frac{1}{N} \quad \text{as} \quad t \to \infty$$

w.p. one (almost surely)

**Proof**

$$\frac{T_n}{n} = \frac{2(2t)}{n} \to N \quad \text{a.s.}$$

By definition,

$$t_{NCE} \leq t < t_{NCE+1}$$

$$\frac{T_{NCE}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{NCE+1}}{N(t)} \to \frac{N(t)+\varepsilon}{N(t)} = \frac{1}{N}$$

Thus, $\frac{N(t)}{t} \to \frac{1}{N}$

**Next goal** was to show

$$\frac{t}{N(t)} \to \frac{1}{N}$$

**Example** let $Y_t = \begin{cases} c + t & \text{if} \quad U \leq \frac{te}{C} \\ c & \text{otherwise} \end{cases}$

$U$ uniform on $[0, 1]$

$$Y_t \to c \quad \text{w.p.} \quad \frac{1}{t} \quad \text{as} \quad t \to \infty$$

$$E[Y_t] = (c + t) \cdot \frac{1}{t} + c \cdot (1 - \frac{1}{t}) \to c + 1$$

Thus, $Y_t \to c$ does not imply $EY_t \to c$.
\[ EN(c) + 1 \geq \frac{c}{\nu} \]

\[ \lim_{t \to \infty} \inf \frac{EN(c)}{t} = \frac{1}{\nu} \quad (1) \]

\[ \overline{t_i} = \min(t_{ij}, m) \text{ truncated interarrival time} \]

\[ \overline{t_n} = \overline{t_1} + \ldots + \overline{t_n} \]

\[ N_m(t) = \overline{t_t} \]

\[ EN(c) \leq EN(t) + 1 \leq \frac{t + m}{\nu} \]

\[ \limsup_{t \to \infty} \frac{EN(c)}{t} \leq \frac{1}{\nu} \quad (8) \]

\[ \lim_{t \to \infty} \frac{E(N_c)}{t} = \frac{1}{\nu} \quad (8) \]

\[ 1 + 2 = \frac{E(N_c)}{t} = \frac{1}{\nu} \]

**Rewards**

Let \( r_i \) be rewards at renewal \( i \)

Assume \( \{r_i\} \) are i.i.d.

\[ R(c) = \sum_{i=1}^{N(c)} r_i \]

\[ \frac{R(c)}{t} = \frac{N(c)}{t} \left( \frac{1}{N(c)} \sum_{i=1}^{N(c)} r_i \right) \to \frac{EC_c}{E_t(c)} \]
Consider an orbiting queue

\[ X_s = \# \text{ Customers at time } t \]

\[ L = \lim_{t \to \infty} \frac{1}{t} \int_0^t X_s \, ds \]

\[ W = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^n W_m \]

The customer \( m \) spends in system

\[ \lambda_m = \lim_{t \to \infty} \frac{N_m(t)}{t} \text{ average arrival rate} \]

Little's result

\[ L = \lambda_m W \]

**M/M/1 Queue**

\[ U_t = \text{ sum of remaining service times in system (workload)} \]

Suppose a customer \( w \) has \( Y \) units of service time remaining pay \( s \) to customer \( y \)

Let \( Y_g \) = Average total payment by each customer

Average work load satisfies

\[ V = \lambda Y \]

\[ Y = E[S_i] \cdot W_q + E \left[ \frac{S_i^2}{2} \right] \]

\[ Y = \lambda E[S_i] \cdot W_q + E \left[ \frac{S_i^2}{2} \right] \]

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\[ W = W_q \]

\[ W_a = E[S_i] \cdot W_q + E \left[ \frac{S_i^2}{2} \right] \]

\[ W_q = \frac{\lambda E[S_i]^2}{2(1 - \lambda E[S_i])} \]

\[ W = W_a + E[S_i] \]

\[ L = \frac{W}{\lambda} \]