Discrete time chain:
\[ \{X_n\} \text{ be a m.c. strecth wit } X_0 \sim \pi \]
\[ \{Y_m\} \equiv Y_m = X_{n-m} \text{ for } 0 \leq n \leq m \]
then \( Y_m \) is a m.c.
\[ r(i,j) = \frac{\pi(i) p(j,i)}{\pi(i)} \]

Continuous time Chain
\[ \{X(t)\} \text{ be a c.t. m.c. with } X_0 \sim \pi \]
Fix \( t \).
\[ \{Y_s\} : Y_s = X_{t-s} \text{ for } 0 \leq s \leq t \]
\( Y_s \) is a m.c.
with transition probability
\[ \gamma(t, i,j) = \frac{\pi(i) P_t(j,i)}{\pi(i)} \]
\[ \Phi(t, i,j) = \frac{\pi(i) Q_t(j,i)}{\pi(i)} \]

Special case: Birth Death chain
\[ \pi(i) q(i,j) = \pi(j) q(j,i) \]
 Balance equation
\[ \Rightarrow \Phi(i,j) = q(i,j) \]
M/M/1 queue is a B. D. chain.

M/M/1 queue running in reverse time has the same transition probs as it does in Forward time.

"arrival of Fwd" \leftrightarrow "departure of reverse" chain

"departure of Fwd" \leftrightarrow "arrival on the reverse" chain

Arrivals of the Forward chain are Poisson \implies Departures of the reverse chain are Poisson.

Reverse chain has identical transition probabilities as the Forward chain.
Thus the departures of Forward chain (in steady-state) are Poisson.

Same argument applies to M/M/l s as well queue.

\[ \rightarrow [ ] \rightarrow [ ] \rightarrow \]

**Theorem** Let \( N(t) \) be the number of departures between time 0 and \( t \) of the M/M/l s start from \( 0 \) stationary distribution \( X(0) \).
Then \( \langle N(t) \rangle : 0 \leq t < \infty \) and \( K(t) \) are independent.

\[ \begin{array}{c}
0 \\
\xrightarrow{t} \\
\xrightarrow{\infty} \\
\text{departures}
\end{array} \]

**Proof**

Reverse time
\[ Y(t+s) = X(t-s) \]

- departures before \( t \) of \( \langle X \rangle \)
- become arrivals after \( t \) of \( \langle Y \rangle \)

- Obviously arrivals after \( t \) are independent of \( Y(t) \)
- But \( X(t) = Y(t) \)
- Thus departures before \( t \) of forward chain are independent of \( K(t) \)

\[ \langle Y \rangle \text{ has poison arrivals independent of current state.} \]

**Tandem \( M/M/1 \)**

\[ \begin{array}{c}
\lambda \\
\xrightarrow{N_1^1} \\
\xrightarrow{N_1} \\
\xrightarrow{N_2} \\
\lambda
\end{array} \]

\( \lambda < \lambda_1 \)
\( \lambda < \lambda_2 \)

let \( \tau \) be large,

\[ P(\text{\# servers at time } \tau = m) = \left( \frac{\lambda}{\lambda_1} \right)^m \left( 1 - \frac{\lambda}{\lambda_1} \right) \]
The departure process from queue 1 is Poisson rate $\lambda$

$$P(N_{e}^1 = n) = \left( \frac{\lambda}{N_{e}} \right)^n (1 - \frac{\lambda}{N_{e}})$$

By reversibility, the departure process from queue 1 up until time $t$ is independent of $N_{e}^1$.

The departure process from queue 1 up until time $t$ will determine $N_{e}^2$

Thus $N_{e}^1$ and $N_{e}^2$ are independent.

$$P(N_{e}^1 = m, N_{e}^2 = n) = \left( \frac{\lambda}{n_1} \right)^m (1 - \frac{\lambda}{n_1}) \left( \frac{\lambda}{n_2} \right)^n (1 - \frac{\lambda}{n_2})$$

Yes this has product form: $(1 - \frac{\lambda}{n_2}) \cdot \left( \frac{\lambda}{n_3} \right)^r \cdot \left( 1 - \frac{\lambda}{n_3} \right)$
\[ \lambda \xrightarrow{\text{delay}} N_{t+1} \xrightarrow{p} \] 

In steady state departures in past independent or \( N_t \). Combined therefore \( n \) arrivals will be independent or \( N_t \)

Thus queue is \( M|M|1 \)

Limit \( \epsilon \to 0 \)

\[ \lambda \xrightarrow{\text{delay}} N_{t+1} \xrightarrow{p} \]

\[ r = \lambda + rp \]

\[ r = \frac{\lambda}{1-p} \]

If \( \frac{\lambda}{1-p} < N_t \) then we have invariant distribution

\[ p(C_{N_t} = n) = \left( \frac{r}{N_t} \right)^n (1 - \frac{r}{N_t}) \]

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**General story**

Suppose we have an arbitrary network of \( k \) stations.

\[ p(i,j) = \text{prob. customer finishing station } i \text{ goes to station } j \]

\[ q(i) = 1 - \sum_j p(i,j) \]  

Chance customer exits
Let's suppose there is a sequence of stations \( j_0, j_1, \ldots, j_n \) with \( p(j_m, j_{m+1}) > 0 \) \( q(j_n) = 0 \) means that for any station, you have a way out (open queueing network).

Consider a vector of rates into each station \( \lambda \).

\[ \dot{\lambda} = \lambda (I - p) \]

\[ \dot{\lambda} = \frac{\lambda}{ \lambda (I - p) } \quad \text{can show algebraically that } \dot{\lambda} \text{ makes this system} \]

inveteble.

If \( r_j < \nu_j \) for \( j \) then for \( K \) \( M/M/1 \) stations

\[ \prod (\nu_1, \ldots, \nu_K) = \prod_{j=1}^{K} \left( \frac{r_j}{\nu_j} \right)^{n_j} (1 - \frac{r_j}{\nu_j}) \]

Suppose departure rate from station \( i \) depends on \( n_i \), the number of customers at the station. \( E_i (n) \) the rate from station \( i \) when \( n \) customers present.

Example: \( M/M/1 \) queue

\[ E_i(n) = \nu_i \min (n, s) \]

So this is describable with a birth death chain

\[ E_i (1) \quad E_i (2) \quad E_i (3) \]

\[ \rho_i \quad \rho_i \quad \rho_i \quad \rho_i \]

Let \( \psi_i (n) = \prod_{m=1}^{n} E_i (m) \)
\[ \sum_{\eta=0}^{\infty} c_j \frac{r_j^n}{\psi_j(n)} = 1 \quad \text{For station } j \]

If queues are stable:

\[ \Pi(n_1, \ldots, n_K) = \prod_{j=1}^{K} \frac{c_j r_j^n}{\psi_j(n_j)} \]

Aside:

\[ \Pi(0) = c \]
\[ \Pi(1) = \frac{c}{e_1 c_0} \Pi(0) \]
\[ \Pi(2) = \frac{c}{e_1 c_0} \cdot \frac{c}{e_2 c_1} \Pi(0) \]
\[ = \frac{c^2}{e_1 c_0 \cdot e_2 c_1} \Pi(0) \]

Closed Queuing Network:

No exogenous arrivals.
No exits.

\[ \text{It turns out these networks have a product form distribution.} \]

\[ \rho = \frac{\rho^*}{PA} \]

Routing matrix