1. Durrett 8.46 pp 207

2. Suppose you manage a network consisting of 3 stations, and each station is a M/M/1 queue. Customers arrive to the network at rate \( \lambda \), and all new arrivals to the network arrive to station 1. Customers who finish service at station 1 go to station 2 with probability \( p \) and go to station 3 with probability \( 1 - p \). Customers who finish service at station 2 return to station 1 with probability \( q \) and exit the system with probability \( 1 - q \). Customers who finish service at station 3 always exit the system. The service rates at the stations 1, 2, and 3 are \( \mu_1, \mu_2, \) and \( \mu_3 \) respectively. (i) Under what conditions will the network be stable? (ii) What is the invariant distribution? (iii) Suppose each unit of service rate costs \$1 and you have \$C to spend. (Thus your constraint is \( \mu_1 + \mu_2 + \mu_3 \leq C \)). What choice of service rates minimizes the average number of customers queued in the system in steady state.

Hints: Write an expression for the expected number of customers in the system as a function of \( \mu_1, \mu_2 \) and \( \mu_3 \). Call this expression \( f(\mu) \). You will notice that increasing any one of the \( \mu_i \)'s decreases \( f(\mu) \). Therefore the solution to your constrained minimization problem (\( \text{min} f(\mu) \) subject to \( \mu_1 + \mu_2 + \mu_3 \leq C \)) must occur on the boundary \( (\mu_1 + \mu_2 + \mu_3 = C) \). (If the constraint weren’t tight, you could improve your objective by increasing one of the \( \mu \)'s.) To find the optimal point, we therefore need to use the technique of Lagrange multipliers.

Let’s briefly give some intuition as to why the technique works. Imagine we are taking a walk along the contour or “hiking path” formed by \( \mu_1 + \mu_2 + \mu_3 = C \) and \( f(\mu) \) is the height of the ground at point \( \mu \). The direction of steepest descent of the function \( f(\mu) \) occurs along its gradient (the vector of partial derivatives). Imagine we are at a point on the “hiking path” where the gradient of \( f(\mu) \) is perpendicular to the direction of the path. At that point the path is not climbing or descending in either direction. The point must be a local minimum or maximum along the path. In contrast, if the direction of the path were closely aligned to the gradient direction, the path would be descending. The position of the path itself is also described by a function (for us it’s \( g(\mu) := \mu_1 + \mu_2 + \mu_3 - C = 0 \)). You can verify that the direction of the path at any point \( \mu \) is perpendicular to the gradient of \( g(\mu) \)

Combining our observations, we see that a local maximum or minimum along our hiking path occurs where the gradients of \( f(\mu) \) and \( g(\mu) \) point in the same direction. This occurs when

\[
\frac{d}{d\mu_i} f(\mu) + \gamma \frac{d}{d\mu_i} g(\mu) = 0
\]

for some constant \( \gamma \) and all \( \mu_i \). To solve our problem write the above equation for \( \mu_i = \mu_1, \mu_2 \), and \( \mu_3 \). These three equations, along with the equation \( g(\mu) = 0 \) should be sufficient to solve for the 4 unknowns: \( \mu_1, \mu_2, \mu_3, \) and \( \gamma \).

3. Suppose you manage a call center with Poisson arrivals of rate 3 per minute. The call center is divided into two stations: station 1 has \( s_1 \) “screener” operators and station 2 has \( s_2 \) “expert” operators. The service time of a call in each station is exponentially distributed with a mean of 1 minute. Thus, each station is an M/M/\( s_i \) queue, and each station has an unlimited “waiting room” for calls to queue while waiting for service. All calls arrive first to station 1. After finishing service at station 1, the calls are either forwarded to station 2 or exit the call center with equal probability. You can think of the calls that get passed from station 1 to station 2 as difficult calls that the screeners in station 1 forwarded to the experts of station 2. Calls that finish service at station 2 always exit the system. (i) What conditions on \( s_1 \) and \( s_2 \) are required for stability? (ii) What is the invariant distribution of this system? (iii) What is the expected number of customers in the system in steady state? It is ok if your expression for this is not simplified. (iv) Suppose that the staff costs of your center are \( s_1 + 2s_2 \) and you must keep your costs at or below \$11. Pick \( s_1 \) and \( s_2 \) to minimize the average number of customers queued in the system in steady state, subject to your budget constraint.

Hints: Argue that the only allocations that you need to compare are \( (s_1, s_2) = (5, 3) \) and \( (7, 2) \). This is because other assignments either make the system unstable, or are obviously poorer performing than either \( (5, 3) \) and \( (7, 2) \). For instance \( (6, 2) \) must perform less well than \( (7, 2) \) and \( (3, 4) \) is unstable because the service rate of the first station is not larger than the incoming rate. You might want to use matlab or excel to help with the calculations.

4. Durrett 5.16 pp 236

5. Durrett 5.18 pp 237