1. Suppose $X, Y, Z$ are pairwise Jointly Gaussian (JG). In other words the pairs $\{X, Y\}$, $\{Y, Z\}$, and $\{X, Z\}$ are JG. Must the collection $\{X, Y, Z\}$ be JG? If so, prove it. If not, provide a counterexample.

2. (a) The conditional variance of $X$ given $Y$ is defined by

$$Var(X|Y) = E[(X - E(X|Y))^2|Y].$$

Prove that

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$

(b) A mouse is trapped in the center of a maze. There are three paths that begin at the center. The first path takes the mouse to safety after 3 hours of travel. The second and third path return the mouse to the center of the maze after 4 and 7 hours of travel respectively. Assume the mouse is equally likely to choose any of the three paths at any time. Find the variance of $X$, the total time the mouse spends in the maze.

3. Suppose that we are designing a digital receiver. The sender will transmit a single bit $H$ that takes the value 0 or 1 with probabilities $P_0$ and $P_1$ respectively. The receiver does not get to see $H$ directly, but instead observes a continuously distributed random vector $\hat{Y}$. $\hat{Y}$ depends on both $H$ as well as random noise that is introduced in the channel that connects the transmitter to the receiver. Let $\hat{H}(\hat{y})$ be the receiver’s guess of what the original bit was given that the receiver observes $\hat{Y} = \hat{y}$. We would like to maximize the probability of the receiver making the correct guess. Therefore, we use the rule

$$P_{H|\hat{Y}}(0|\hat{y}) \begin{cases} \hat{H}=0 \\ \hat{H}=1 \end{cases} P_{H|\hat{Y}}(1|\hat{y})$$

where the notation $P_{H|\hat{Y}}(0|\hat{y})$ means $P(H = 0|\hat{Y} = \hat{y})$ and the $\hat{H}=0$ notation means that the receiver picks $\hat{H} = 1$ if the left side is larger, and picks $\hat{H} = 0$ otherwise. Rule (1) is known as a maximum a–posterior probability (MAP) rule. It turns out we can simplify (1) in the following way. First, we make use of Bayes rule to write

$$\lim_{\epsilon \to 0} \frac{P(H = 0, \hat{Y} \in B(\hat{y}, \epsilon))}{P(\hat{Y} \in B(\hat{y}, \epsilon))} \begin{cases} \hat{H}=0 \\ \hat{H}=1 \end{cases} \lim_{\epsilon \to 0} \frac{P(H = 1, \hat{Y} \in B(\hat{y}, \epsilon))}{P(\hat{Y} = B(\hat{y}, \epsilon))}.$$ 

where $B(\hat{y}, \epsilon)$ is a ball centered at $\hat{y}$ with radius $\epsilon$. We apply Bayes rule again to write

$$\lim_{\epsilon \to 0} \frac{P_0 P(\hat{Y} \in B(\hat{y}, \epsilon)|H = 0)}{P(\hat{Y} = B(\hat{y}, \epsilon))} \begin{cases} \hat{H}=0 \\ \hat{H}=1 \end{cases} \lim_{\epsilon \to 0} \frac{P_1 P(\hat{Y} \in B(\hat{y}, \epsilon)|H = 1)}{P(\hat{Y} = B(\hat{y}, \epsilon))}.$$ 

The left hand side of (2) is known as the likelihood ratio. Oftentimes, we do not know the “prior” probabilities, that is the probabilities that the sender sends a 1 or 0. In that case, we can assume that $P_0 = P_1$. When we assume $P_0 = P_1$, rule (2) is called a maximum likelihood(ML) test. When we use the values of the priors $P_0$ and $P_1$ in rule (2), then it is MAP rule.

(a) Suppose that the sender sends a signal of size $X = a$ to represent the bit $H = 0$ and a signal of $X = b$ to represent the bit $H = 1$. Suppose that the receiver receives the signal $Y = X + Z$
where $Z \sim N(0, \sigma^2)$. What is the MAP rule for determining $\hat{H}$? Hint: Start with (2). To simplify your final expression as much as possible, take the log of both sides to eliminate exponential functions. Also put all terms dependent on $y$ on the left side of your final expression, and all terms not dependent on $y$ on the right side of your expression.

(b) In part (a), suppose $\sigma = 1$, $a = 0$, $b = 4$, and $P_0 = 0.6$. What is the probability of your MAP detection rule making an error?

(c) Suppose the sender sends a vector valued signal $\vec{X} = \vec{a}$ to represent a 0 and $\vec{X} = \vec{b}$ to represent a 1. Suppose that the receiver receives the signal

$$\vec{Y} = \vec{X} + \vec{Z}$$

where $Z \sim N(0, \sigma^2 I)$. What is the MAP rule for determining $\hat{H}$? Simplify your answer as much as possible.

(d) Repeat (c) but now suppose that $Z \sim N(0, K_z)$, where $K_z$ is non-singular. Hint: There exists a nonsingular matrix $A$ such that $K_z = AA^T$. The signal $\vec{W} := A^{-1}\vec{Y}$ contains all of the information that $\vec{Y}$ does because we can recover $\vec{Y}$ from $\vec{W}$. Thus you can construct your detector using $\vec{W}$.

(e) Now suppose that $K_z$ is singular, and in particular $\vec{v}^T K_z \vec{v} = 0$. Also suppose that $\vec{v}^T (\vec{a} - \vec{b}) \neq 0$. What is the MAP rule? Hint: consider the signal $\vec{v}^T \vec{Y}$. 