1 Decision Analysis: A Framework for Rational Decision-Making

While most methods examined up to now assume accurate information, in reality many decisions involve uncertainty. Decision analysis was created to incorporate this uncertainty into the decision. Furthermore, sometimes we have the option to pay for more information to reduce that uncertainty. Using decision analysis, we can figure out when paying for more information is worthwhile.

Examples of the need for rational decision-making under uncertainty are:
- How should we invest given changes in interest rates?
- How deep and where should oil companies drill?
- Should pharmaceutical companies continue current R&D investments?

We will step through an example from the text, Chapter 15: the Goforbroke Oil Company. They own some land, but have not looked for oil there yet. Then someone offers them $90,000 for the land. Should they take the money and sell the land, or drill for oil? It costs them $100,000 to drill, and if they strike oil the payoff is $800,000. If they drill and find that the well is dry, the payoff is $0. These options can be summarized in the payoff table, Figure 1.

<table>
<thead>
<tr>
<th></th>
<th>oil</th>
<th>dry well</th>
</tr>
</thead>
<tbody>
<tr>
<td>drill</td>
<td>700</td>
<td>-100</td>
</tr>
<tr>
<td>sell</td>
<td>90</td>
<td>90</td>
</tr>
</tbody>
</table>

Say we are running the oil company. Our objective then is to maximize our own gain, given the payoff distribution that depends on the state. We need to account for all the alternatives, and the uncertain "state of nature.” Note that we are playing against nature, and not against a rational opponent who works against you, as was the case in game theory. Here, nature is not necessarily your adversary; nature is not rational, and it is not greedy.
2 Strategy Options

Here are three strategy options for the decision maker:

(a) **Maximize the minimum payoff.** Consider this a game against nature, and we seek the best guaranteed outcome. This doesn’t take into account the likelihood of events, and is a very conservative strategy. This is also known as the maximin payoff criterion, p. 683.

(b) **Maximum likelihood.** We identify the most likely outcome, and choose the decision that maximizes the payoff for that most likely outcome. But what if the most likely state is still unlikely, or there is a huge payoff difference among states? This method ignores much relevant information.

(c) **Bayes’ Decision Rule.** Take the best expected payoff. This is the most commonly used strategy. Using the best available estimates of the probabilities of the respective states of nature, calculate the expected value of the payoff for each of the possible decision alternatives. Choose the decision alternative with the maximum expected payoff.

For our example, say we think the probabilities are $P(\text{oil}) = \frac{1}{4}$, and $P(\text{dry}) = \frac{3}{4}$. Then

\[
E[\text{Payoff(Drill)}] = \frac{1}{4}(700) + \frac{3}{4}(-100) = 100
\]

\[
E[\text{Payoff(Sell)}] = \frac{1}{4}(90) + \frac{3}{4}(90) = 90
\]

Note however that probabilities can be subjective. Do you trust an expert? Since there is uncertainty in the \textit{a priori} values, sensitivity analysis on the probabilities is a good idea. Letting the variable $p$ be the probability of finding oil, we seek the crossover point where the decisions to drill and to sell have equal expected payoffs.

\[
E[\text{Payoff(Drill)}] = 700p + -100(1 - p) = 800p - 100 = 90
\]

\[
E[\text{Payoff(Sell)}] = 90
\]

Here $p_{\text{crossover}} = 0.2375$, so if $p > 0.2375$ drill, and if $p < 0.2375$ sell the land.

A decision plot for this scenario is shown in Figure 1. Note that at every node, all the probability branches emanating from it sum to one. If there were more options, there would be more lines on the plot. If there were more uncertainties, there would be more dimensions.

- **Calculating prior probabilities** can be a complex undertaking, and there are courses devoted to this subject. One way to go about it is to seek the decision maker’s personally indifferent buying price. An example of a person’s personally indifferent buying price for the likelihood of an event could be for, say, the probability that a Republican will win the 2008 presidential election. Given a choice between the outcomes \{GOP president\} or \{heads|50-50 coin\}, which would you choose?
Figure 1: This shows two ways to evaluate the decision to sell or drill. On the left is a graph from a sensitivity analysis of the expected payoff versus the probability of striking oil. On the right is a decision tree for computing conditional probabilities of outcomes.

How about between the outcomes {GOP president} or {heads|40-60 coin}, where heads are known to come up only 40% of the time? If you choose the latter, you are saying you expect the odds of the GOP winning to be less than 40%. If you then choose {GOP president} over {heads|35-65 coin}, you have identified your indifferent buying price for this issue as between 35% and 40%.

Financial markets are assumed to be perfect, where you can buy or sell at will, and people are rational. You can use market predictions and monetary bets to infer probabilities. In reality, values and probabilities come from a combination of expert opinion and market data.

3 The Value of Information

Returning to the Goforbroke Oil Co. problem, we would like to know if we can improve our prior probability estimates using some prior experiment. Geologists will conduct a seismic study of the land for $30,000; is it worth it to pay for this test? Their test has a binary outcome, either "likely" or "unlikely".

\[ P[\text{Unlikely}|\text{there really is oil}] = 0.4 \]  
\[ P[\text{Likely}|\text{oil}] = 0.6 \]  
\[ P[\text{Unlikely}|\text{it really is dry}] = 0.8 \]  
\[ P[\text{Likely}|\text{dry}] = 0.2 \]  

These probabilities are based on past experience.

Bayes’ theorem states that: 
\[ P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)} \]

In our example, let A be the state of the well, dry or having oil. Let B be the test result, likely or unlikely. Then

\[ P[\text{oil}|\text{unlikely}] = \frac{(0.4)(0.25)}{(0.4)(0.25) + (0.8)(0.75)} = \frac{1}{7} \]  

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So the posterior probability of oil, given the outcome of the test is unlikely, is 1/7. Similarly

\[
P[\text{dry} | \text{unlikely}] = \frac{6}{7} \quad (3.6)
\]
\[
P[\text{oil} | \text{likely}] = \frac{1}{2} \quad (3.7)
\]
\[
P[\text{dry} | \text{likely}] = \frac{1}{2} \quad (3.8)
\]
\[
P(\text{likely}) = 0.15 + 0.15 = 0.3 \quad (3.9)
\]
\[
P(\text{unlikely}) = 0.1 + 0.6 = 0.7 \quad (3.10)
\]
\[
P[\text{oil} | \text{likely}] = \frac{0.15}{0.15 + 0.15} = \frac{1}{2} \quad (3.11)
\]
\[
P[\text{oil} | \text{unlikely}] = \frac{0.1}{0.1 + 0.6} = \frac{1}{7} \quad (3.12)
\]

The new expected payoffs become:

\[
E[\text{Payoff (drill | likely)}] = (0.5)(700) + (0.5)(-100) - 30 = 270 \quad (3.14)
\]
\[
E[\text{Payoff (sell | likely)}] = (0.5)(90) + (0.5)(90) - 30 = 60 \quad (3.15)
\]
\[
E[\text{Payoff (drill | unlikely)}] = \frac{1}{7}(700) + \frac{6}{7}(-100) - 30 = -15.7 \quad (3.16)
\]
\[
E[\text{Payoff (sell | unlikely)}] = \frac{1}{7}(90) + \frac{6}{7}(90) - 30 = 60 \quad (3.17)
\]

The decision is to drill if the test comes up likely, because its payoff is 270 > 60. If the test comes up unlikely, the decision is to sell, because 60 > -15.7.

The upper bound on the benefit of doing the seismic test is the case where it gives us perfect information on the state of nature. The expected payoff with perfect information is (0.25)(700) + (0.75)(90) = 242.5, using the prior probabilities from Section 2. (Note that ALL of these probabilities are guesses based on historical data, so this is "perfect information" relative to those guesses.)

Recall that the expected payoff without the information from the seismic test is 100. So the value of knowing the true state of nature is 242.5 - 100 = 142.5, or $142,500; if the seismic test gave perfect information, its value is $142,500 > $30,000, and is well worth paying for.

Figure 2 shows the final decision tree for the problem. The actual seismic test does not give perfect information, so to calculate the value of performing the test, work backwards from the outer branches of the tree inwards. At each node, choose the branch whose expected payoff, P*value, is the highest. At the branch "unfavorable vs. favorable", the values are 60 for the unlikely subtree, and 270 for the likely subtree. P(likely subtree) = 0.3, P(unlikely subtree) = 0.7, and

\[
E[\text{Payoff with experiment}] = (0.7)(60) + (0.3)(270) = 123 \quad (3.18)
\]
The expected payoff without doing the experiment as before is 100; 123 > 100, so the decision under the Bayes’ decision rule strategy will be to do the seismic test.

4 A Crash Course in Utility Theory

People’s decisions are affected by their utility values, or in other words how important they perceive different payoff values to be depending on the risk of not attaining those payoffs. In particular most people have a decreasing marginal utility for money. So the more money you have, the less another increment of money means to you. Such a utility function, $U$, is increasing and concave.

For instance, most people would prefer a sure $500 to a 50-50 chance at $100. If you prefer the $500 payoff, then you are said to be ”risk-averse.” An example of a risk-averse utility curve is shown in Figure 2. The standard utility function formula used to generate such a curve is

$$U(M) = R(1 - e^{-M/R})$$

where $M$ is the amount of money, $R$ is the risk tolerance, and $U$ is the utility value.

We can use utility values instead of actual values in decision analysis problems. The same underlying machinery still holds, such as Bayes’ rule, decision trees, and so on. Then you can work out what the final utility would be, instead of the final value.

See the example on page 711 of the text. Two alternatives are given: would you prefer (1) $0, or (2) $700 with probability $p$, and -$130 with probability $1-p$? This is another way of asking, what is your breakeven probability $p^*$? In this example, solving for $p$ gives $p^* = \frac{4}{5}$. To generate a utility curve, solve for a number of breakeven probabilities and plot their results.

–End of May 10 lecture.–