1 Main Ideas

The three main ideas of inventory theory:

1. It costs money to store goods
2. It costs money not to have goods ready to sell
3. It costs money to order goods

How do we minimize total cost and meet demand? We can model stack level and optimize the policy of reordering (the next step is automating policy).

\[ C(Z) = \text{Cost to order } z \text{ units} \]
\[ C(Z) = \begin{cases} 0 & z = 0 \\ K + cz & z > 0 \end{cases} \quad \text{where } K = \text{Setup cost}, c = \text{Unit cost} \]

\[ h(x) = \text{Holding/storage cost per unit of time} \]

Sometimes there are other factors to include such as revenue and salvage costs.

There are two main decision classes:
1. Continuous review
2. Periodic review

2 Basic Model: Economic Order Quantity

- Product is sold at a known constant rate
- \( K \) = Setup cost
- \( c \) = Unit cost
- \( h \) = Holding cost per unit per unit of time
- \( d \) = Known constant demand rate
- Continuous review
- Shortages not allowed
- Fixed lead time from order to become available

Our optimal policy to have an order arrive when inventory hits zero. Thus make sure the order takes the delivery time into account. Order up to "reorder point" \( Q \) (time to order = demand rate \times \) lead time).

Minimize cost per unit time so for each cycle: Order \( Q \) cost \( K + cQ \)

- Average inventory level = \( Q/2 \)
- Holding cost per unit of time = \( \frac{hQ^2}{2} \)
- Length of cycle = \( Q/d \)

\( \therefore \) Holding cost/cycle = \( \frac{hQ^2}{2d} \) \( \therefore \) Total cost = \( K + cQ + \frac{hQ^2}{2d} \) per cycle

Cost/unit time: \( T = \frac{K + cQ + \frac{hQ^2}{2d}}{Q/d} = \frac{4K}{Q} + dc + \frac{hQ}{2} \)

Minimizing \( T \) with respect to \( Q \), \( \frac{dT}{dQ} = \frac{h}{2} - \frac{Kd}{Q^2} \)

Equating it to zero, we have, \( \frac{h}{2} = \frac{Kd}{Q^2} \) \( \therefore \) \( Q^* = \sqrt{\frac{2dK}{h}} \)

Optimal cycle time = \( \frac{Q^*}{d} = \sqrt{\frac{2K}{hd}} \)

Note that this value is independent of \( c \) (unit cost). This is because under this policy we always get to meet the demand, so any value of \( c \) is acceptable.
What if we allows shortages? That is to say, what if you don’t have everything to sell so that there are times when the inventory dips below zero.

- $p$: Shortage cost per unit of time
- $S$: Inventory after adding $Q$ units
- $Q - S$: Shortage just prior to order

Holding cost/cycle = $\frac{hs}{2} \frac{S}{d} = \frac{hS^2}{2d}$

Shortage cost = $\frac{p(Q-S)Q-S}{2d} = \frac{(Q-S)^2}{2d}$

Total cost/unit time: Divide by $\frac{Q}{d} = \frac{dK}{Q} + dc + \frac{hs^2}{2Q} + \frac{p(Q-S)^2}{2Q}$

Differentiating with respect to $S$ and $Q$ gives,

$S^* = \sqrt{\frac{2dK}{h}} \sqrt{\frac{p}{p+h}}$ ; $Q^* = \sqrt{\frac{2dK}{h}} \sqrt{\frac{p+h}{p}}$ ; $t^* = \frac{Q^*}{d}$ = $\sqrt{\frac{2dK}{dh}} \sqrt{\frac{p+h}{p}}$

Maximum shortage = $Q^* - S^* = \sqrt{\frac{2pK}{p}} \sqrt{\frac{h}{p+h}}$

Fraction of time without shortage = $\frac{p}{p+h} = \frac{S^*/d}{Q^*/d}$

2.1 "Just in Time" (JIT) Inventory

This strategy was made famous by Japanese manufacturer Toyota. The main idea is that if we can reduce setup costs to zero, then it becomes optimal to carry zero inventory.
3 Quantity Discounts

The main idea of a quantity discount is that unit cost decreases the larger the order. For example when using discrete price levels it might cost $100 for up to 150 orders, $75 up to 500, etc. To figure out the optimal solution we can still use the main ideas previously discussed and calculate optimal $Q^*_j$ for each cost

- If feasible, keep that and corresponding $T^*_j$
- If infeasible, take nearest $Q^*_j$ and $T^*_j$
- Take $Q^*_j$ with minimum $T^*_j$ value

4 Deterministic Periodic Review

The basic idea of deterministic periodic review is that there is a different demand for each period but that demand is still known. We express this using $r_i = \text{demand per period } i$. The optimal policy remains to only produce when inventory is zero. This reduces the problem to a (simple) dynamic program! We say that the dynamic programming problem is easier to solve than normal because we know that the optimal policy will let the inventory hit zero.

\[
C_i = \text{Total cost of optimal policy for } i, i+1, ..., n \text{ when period } i \text{ starts with zero inventory.}
\]

\[
C_i = \min_{j=1,...,n} \{C_{j+1} + K + h[r_{i+1} + 2r_{i+2} + ... + (j-i)r_j]\}
\]

\[j = \text{period when inventory next hit zero after } i.\]

\[C_{n+1} = 0 \text{ (efficient PD since state space is small)}\]

5 Stochastic Inventory

Usually demand is uncertain and we must deal with risk and probabilistic outcomes. If we aim to let stock hit zero we may lose some sales because we will not have any actual stock left. Thus we may adopt a policy of keeping some safety stock around. When stock goes below $R$, order $Q$ (or up to $S$). The exact value for $R$ often depends on a management policy and often depends on the probability of the stock levels dropping below zero.

Select $R$ and $Q$ → Full argument p. 871

Use expected demand $d$

\[
Q = \sqrt{\frac{2dK}{h} \sqrt{\frac{\nu + h}{p}}} \\
D = \text{demand during lead time} \\
P(D \leq R) = L
\]

e.g. $R = \mu + K_{1-L}\sigma$ for normal distribution
6 Stochastic Perishable Products

These are items that we might consider to have a “shelf-life”. This might include things like newspapers, produce, fashion items or even things like cars which have a limited model life. The key is that each product only lasts a certain period of time or starts to loss value after a certain period of time. With these products there is often a salvage value.

$I = \text{Initial inventory}$

$Q = \text{Order quantity}$

$S = I + Q = \text{Initial stock after ordering}$

$D = \text{Stochastic demand}$

$K, c, p$ as before.

$h = \text{Holding cost} = \text{Storage cost} - \text{Salvage cost}$

Expected cost

$\overline{C}(S) = K + c(S - I) + \int_{S}^{\infty} p(x - S)f(x)dx + \int_{0}^{S} h(S - x)f(x)dx$ if order;

(or)

$\int_{S}^{\infty} p(x - S)f(x)dx + \int_{0}^{S} h(S - x)f(x)dx$ if don’t order

Leads to an $(s, S)$ policy

If $I < S^*$ order up to $S^*$

If $I > S^*$ do not order