Homework 4
ISM 206, Fall 2010
Due: Tuesday, Nov 23, in class.

Reading: Chapters 14,15,16 Hillier and Lieberman
Problems identified by number are taken from Hillier and Lieberman 8th edition. (Note this may not be the edition in the library).

1. Problem 14.2-7
2. Problem 14.4-4
3. Problem 15.2-5
4. Problem 15.3-10
5. Problem 16.5-8
6. Problem 16.7-1
PROBLEMS

14.2-7. Two politicians soon will be starting their campaigns against each other for a certain political office. Each must now select the main issue she will emphasize as the theme of her campaign. Each has three advantageous issues from which to choose, but the relative effectiveness of each one would depend upon the issue chosen by the opponent. In particular, the estimated increase in the vote for politician 1 (expressed as a percentage of the total vote) resulting from each combination of issues is as follows:

<table>
<thead>
<tr>
<th>Issue for Politician 1</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Issue for Politician 2</td>
<td>1</td>
<td>7</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-5</td>
<td>-3</td>
</tr>
</tbody>
</table>

However, because considerable staff work is required to research and formulate the issue chosen, each politician must make her own choice before learning the opponent’s choice. Which issue should she choose?

(a) For each of the situations described here, formulate this problem as a two-person, zero-sum game, and then determine which issue should be chosen by each politician according to the specified criterion.

(b) The current preferences of the voters are very uncertain, so each additional percent of votes won by one of the politicians has the same value to her. Use the minimax criterion.

(c) A reliable poll has found that the percentage of the voters currently preferring politician 1 (before the issues have been raised) lies between 45 and 50 percent. (Assume a uniform distribution over this range.) Use the concept of dominated strategies, beginning with the strategies for politician 1.

(d) Suppose that the percentage described in part (b) actually were 45 percent. Should politician 1 use the minimax criterion? Explain. Which issue would you recommend? Why?

14.4-1. Reconsider Prob. 14.3-1. Use the graphical procedure described in Sec. 14.4 to determine the optimal mixed strategy for each player according to the minimax criterion. Also give the corresponding value of the game.

14.4-2. Consider the game having the following payoff table.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Player 1</td>
<td>3</td>
</tr>
<tr>
<td>Player 2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Use the graphical procedure described in Sec. 14.4 to determine the value of the game and the optimal mixed strategy for each player according to the minimax criterion. Check your answer for player 2 by constructing his payoff table and applying the graphical procedure directly to this table.

14.4-3. For the game having the following payoff table, use the graphical procedure described in Sec. 14.4 to determine the value of the game and the optimal mixed strategy for each player according to the minimax criterion.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Player 1</td>
<td>4</td>
</tr>
<tr>
<td>Player 2</td>
<td>0</td>
</tr>
</tbody>
</table>

14.4-4. The A. J. Swim Team soon will have an important swim meet with the G. N. Swim Team. Each team has a star swimmer (John and Mark, respectively) who can swim very well in the 100-yard butterfly, backstroke, and breaststroke events. However, the rules prevent them from being used in more than two of these events. Therefore, their coaches now need to decide how to use them to maximum advantage.

Each team will enter three swimmers per event (the maximum allowed). For each event, the following table gives the best time previously achieved by John and Mark as well as the best time for each of the other swimmers who will definitely enter that event.
(Whichever event John or Mark does not swim, his team’s third entry for that event will be slower than the two shown in the table.)

<table>
<thead>
<tr>
<th>A. J. Swim Team</th>
<th>G. N. Swim Team</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Entry</strong></td>
<td><strong>Entry</strong></td>
</tr>
<tr>
<td>1 2 John</td>
<td>1 2 Mark</td>
</tr>
<tr>
<td>Butterfly stroke</td>
<td>1:01.6 59.1 57.5</td>
</tr>
<tr>
<td>Backstroke</td>
<td>1:06.8 1:05.6 1:03.3</td>
</tr>
<tr>
<td>Breaststroke</td>
<td>1:13.9 1:12.5 1:04.7</td>
</tr>
</tbody>
</table>

The points awarded are 5 points for first place, 3 points for second place, 1 point for third place, and none for lower places. Both coaches believe that all swimmers will essentially equal their best times in this meet. Thus, John and Mark each will definitely be entered in two of these three events.

(a) The coaches must submit all their entries before the meet without knowing the entries for the other team, and no changes are permitted later. The outcome of the meet is very uncertain, so each additional point has equal value for the coaches. Formulate this problem as a two-person, zero-sum game. Eliminate dominated strategies, and then use the graphical procedure described in Sec. 14.4 to find the optimal mixed strategy for each team according to the minimax criterion.

(b) The situation and assignment are the same as in part (a), except that both coaches now believe that the A. J. team will win the swim meet if it can win 13 or more points in these three events, but will lose with less than 13 points. [Compare the resulting optimal mixed strategies with those obtained in part (a).]

(c) Now suppose that the coaches submit their entries during the meet event at a time. When submitting her entries for an event, the coach does not know who will be swimming that event for the other team, but he does know who has swam in preceding events. The three key events just discussed are swim in the order listed in the table. Once again, the A. J. team needs 13 points in these events to win the swim meet. Formulate this problem as a two-person, zero-sum game and use the concept of dominated strategies to determine the best strategy for the G. N. team that actually “guarantees” it will win under the assumptions being made.

(d) The situation is the same as in part (c). However, now assume that the coach for the G. N. team does not know about game theory and so may, in fact, choose any of his available strategies that have Mark swimming two events. Use the concept of dominated strategies to determine the best strategies from which the coach for the A. J. team should choose. If this coach knows that the other coach has a tendency to enter Mark in the butterfly and the backstroke more often than in the breaststroke, which strategy should she choose?

14.5.1. Refer to the last paragraph of Sec. 14.5. Suppose that 3 were added to all the entries of Table 14.6 to ensure that the corresponding linear programming models for both players have feasible solutions with $x_3 \geq 0$ and $y_2 \geq 0$. Write out these two models.

14.5.2. Consider the game having the following payoff table:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>1 2 3</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1</td>
<td>1 3 0 4</td>
<td>1 -3 2 -2</td>
</tr>
<tr>
<td>2</td>
<td>2 3 0 4</td>
<td>2 3 0 4</td>
</tr>
<tr>
<td>3</td>
<td>0 4 -1 -3</td>
<td>0 -1 -3</td>
</tr>
</tbody>
</table>

Based on the information given in Sec. 14.5, what are the solutions for these two models? What is the relationship between the value of the linear game and the values of $x_3$ and $y_2$?

14.5-3. Follow the instructions of Prob. 14.5-2 for the game having the following payoff table:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>1 2 3</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1</td>
<td>1 3 0 4</td>
<td>1 -3 2 -2</td>
</tr>
<tr>
<td>2</td>
<td>2 3 0 4</td>
<td>2 3 0 4</td>
</tr>
<tr>
<td>3</td>
<td>0 4 -1 -3</td>
<td>0 -1 -3</td>
</tr>
</tbody>
</table>

14.5-4. Follow the instructions of Prob. 14.5-2 for the game having the following payoff table:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>1 2 3</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1</td>
<td>1 -3 2 -2</td>
<td>1 -3 2 -2</td>
</tr>
<tr>
<td>2</td>
<td>2 3 0 4</td>
<td>2 3 0 4</td>
</tr>
<tr>
<td>3</td>
<td>0 4 -1 -3</td>
<td>0 -1 -3</td>
</tr>
<tr>
<td>4</td>
<td>-4 0 2 2</td>
<td>-4 0 2 2</td>
</tr>
</tbody>
</table>

14.5-5. Section 14.5 presents a general linear programming formulation for finding an optimal mixed strategy for player 1 and player 2. Using Table 14.6, show that the linear programming problem given for player 2 is the dual of the problem given for player 1. (Hint: Remember that a dual variable with a nonpositivity constraint $y_i \leq 0$ can be replaced by $y_i = -y'_i$ with a nonnegativity constraint $y_i \geq 0$.)

14.5-6. Consider the linear programming models for players 1 and 2 given near the end of Sec. 14.5 for variation 3 of the political campaign problem (see Table 14.6). Follow the instructions of Prob. 14.5-5 for these two models.
15.2.2. Jean Clark is the manager of the Midtown Saveway Grocery Store. She now needs to replenish her supply of strawberries. Her regular supplier can provide as many cases as she wants. However, because these strawberries already are very ripe, she will need to sell them tomorrow and then discard any that remain unsold. Jean estimates that she will be able to sell 10, 11, 12, or 13 cases tomorrow. She can purchase the strawberries for $3 per case and sell them for $8 per case. Jean now needs to decide how many cases to purchase.

Jean has checked the store's records on daily sales of strawberries. On this basis, she estimates that the prior probabilities are 0.2, 0.4, 0.3, and 0.1 for being able to sell 10, 11, 12, and 13 cases of strawberries tomorrow.

(a) Develop a decision analysis formulation of this problem by identifying the decision alternatives, the states of nature, and the payoff table.

(b) How many cases of strawberries should Jean purchase if she uses the maximin payoff criterion?

(c) How many cases should be purchased according to the maximum likelihood criterion?

(d) How many cases should be purchased according to Bayes' decision rule?

(e) Jean thinks she has the prior probabilities just about right for selling 10 cases and selling 13 cases, but is uncertain about how to split the prior probabilities for 11 cases and 12 cases. Reapply Bayes' decision rule when the prior probabilities of 11 and 12 cases are (i) 0.2 and 0.5, (ii) 0.3 and 0.4, and (iii) 0.5 and 0.2.

15.2.3. Warren Buffy is an enormously wealthy investor who has built his fortune through his legendary investing acumen. He currently has been offered three major investments and he would like to choose one. The first one is a conservative investment that would perform very well in an improving economy and only suffer a small loss in a worsening economy. The second is a speculative investment that would perform extremely well in an improving economy but would do very badly in a worsening economy. The third is a countercyclical investment that would lose some money in an improving economy but would perform well in a worsening economy.

Warren believes that there are three possible scenarios over the lives of these potential investments: (1) an improving economy, (2) a stable economy, and (3) a worsening economy. He is pessimistic about where the economy is headed, and so has assigned prior probabilities of 0.1, 0.5, and 0.4, respectively, to these three scenarios. He also estimates that his profits under these respective scenarios are those given by the following table:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>220</td>
<td>170</td>
<td>110</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>200</td>
<td>180</td>
<td>150</td>
</tr>
</tbody>
</table>

Which investment should Warren make under each of the following criteria?

(a) Maximin payoff criterion.

(b) Maximum likelihood criterion.

(c) Bayes' decision rule.

15.2.4. Reconsider Prob. 15.2.3. Warren Buffy decides that Bayes' decision rule is his most reliable decision criterion. He believes that 0.1 is just about right as the prior probability of an improving economy, but is quite uncertain about how to split the remaining probabilities between a stable economy and a worsening economy. Therefore, he now wishes to do sensitivity analysis with respect to these latter two prior probabilities.

(a) Reapply Bayes' decision rule when the prior probability of a stable economy is 0.3 and the prior probability of a worsening economy is 0.6.

(b) Reapply Bayes' decision rule when the prior probability of a stable economy is 0.7 and the prior probability of a worsening economy is 0.2.

(c) Graph the expected profit for each of the three investment alternatives versus the prior probability of a stable economy (with the prior probability of an improving economy fixed at 0.1). Use this graph to identify the crossover points where the decision shifts from one investment to another.

(d) Use algebra to solve for the crossover points identified in part (c).

(e) Develop a graph that plots the expected profit (when using Bayes' decision rule) versus the prior probability of a stable economy.

15.2.5. You are given the following payoff table (in units of thousands of dollars) for a decision analysis problem:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>$30 million</td>
<td>$5 million</td>
<td>$10 million</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>$40 million</td>
<td>$10 million</td>
<td>$30 million</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>$10 million</td>
<td>$0</td>
<td>$15 million</td>
</tr>
</tbody>
</table>

(a) Which alternative should be chosen under the maximin payoff criterion?

(b) Which alternative should be chosen under the maximum likelihood criterion?

(c) Which alternative should be chosen under Bayes' decision rule?

(d) Using Bayes' decision rule, do sensitivity analysis graphically with respect to the prior probabilities of states \( S_1 \) and \( S_2 \) (without changing the prior probability of state \( S_3 \)) to determine the crossover point where the decision shifts from one alternative to the other. Then use algebra to calculate this crossover point.

(e) Repeat part (d) for the prior probabilities of states \( S_2 \) and \( S_3 \).

(f) Repeat part (d) for the prior probabilities of states \( S_3 \) and \( S_1 \).

(g) If you feel that the true probabilities of the states of nature are within 10 percent of the given prior probabilities, which alternative would you choose?
does not know which of the two plants this is. The Air Force does have access to the data on the number of spares actually required for the older version, but the supplier has not revealed the production location.

(a) How much money is it worthwhile to pay for perfect information on which plant will produce these engines?

(b) Assume that the cost of the data on the old airplane model is free and that 30 spares were required. You are given that the probability of 30 spares, given a Poisson distribution with mean \( \theta \), is 0.013 for \( \theta = 21 \) and 0.036 for \( \theta = 24 \). Find the optimal action under Bayes’ decision rule.

15.3-8. Vincent Cuomo is the credit manager for the Fine Fabrics Mill. He is currently faced with the question of whether to extend $100,000 credit to a potential new customer, a dress manufacturer. Vincent has three categories for the credit-worthiness of a company: poor risk, average risk, and good risk, but he does not know which category fits this potential customer. Experience indicates that 20 percent of companies similar to this dress manufacturer are poor risks, 50 percent are average risks, and 30 percent are good risks. If credit is extended, the expected profit for poor risks is $-15,000, for average risks $10,000, and for good risks $20,000. If credit is not extended, the dress manufacturer will turn to another mill. Vincent is able to consult a credit-rating organization for a fee of $5,000 per company evaluated. For companies whose actual credit record with the mill turns out to fall into each of the three categories, the following table shows the percentages that were given each of the three possible credit evaluations by the credit-rating organization.

<table>
<thead>
<tr>
<th>Credit Evaluation</th>
<th>Actual Credit Record</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poor</td>
<td>50% 40% 20%</td>
</tr>
<tr>
<td>Average</td>
<td>40% 50% 40%</td>
</tr>
<tr>
<td>Good</td>
<td>10% 10% 10%</td>
</tr>
</tbody>
</table>

(a) Develop a decision analysis formulation of this problem by identifying the decision alternatives, the states of nature, and the payoff table when the credit-rating organization is not used.

(b) Assuming the credit-rating organization is not used, use Bayes’ decision rule to determine which decision alternative should be chosen.

(c) Find EVPL. Does this answer indicate that consideration should be given to conducting the market survey?

(d) Assume now that the market survey is conducted. Find the posterior probabilities of the respective states of nature for each of the two possible predictions from the market survey.

(e) Find the optimal policy regarding whether to conduct the market survey and whether to develop and market the new product.

15.3-9. An athletic league does drug testing of its athletes, 10 percent of whom use drugs. This test, however, is only 95 percent reliable. That is, a drug user will test positive with probability 0.95 and negative with probability 0.05, and a nonuser will test negative with probability 0.95 and positive with probability 0.05.

Develop a probability tree diagram to determine the posterior probability of each of the following outcomes of testing an athlete.

(a) The athlete is a drug user, given that the test is positive.

(b) The athlete is not a drug user, given that the test is positive.

(c) The athlete is a drug user, given that the test is negative.

(d) The athlete is not a drug user, given that the test is negative.

(e) Use the corresponding Excel template to check your answers in the preceding parts.

15.3-10. Management of the Telemore Company is considering developing and marketing a new product. It is estimated to be twice as likely that the product would prove to be successful as unsuccessful. If it were successful, the expected profit would be $1,500,000. If unsuccessful, the expected loss would be $1,800,000. A marketing survey can be conducted at a cost of $300,000 to predict whether the product would be successful. Past experience with such surveys indicates that successful products have been predicted to be successful 80 percent of the time, whereas unsuccessful products have been predicted to be unsuccessful 70 percent of the time.

(a) Develop a decision analysis formulation of this problem by identifying the decision alternatives, the states of nature, and the payoff table when the market survey is not conducted.

(b) Assuming the market survey is conducted, use Bayes’ decision rule to determine which decision alternative should be chosen.

(c) Find EVPL. Does this answer indicate that consideration should be given to conducting the market survey?

(d) Assume now that the market survey is conducted. Find the posterior probabilities of the respective states of nature for each of the two possible predictions from the market survey.

(e) Find the optimal policy regarding whether to conduct the market survey and whether to develop and market the new product.

15.3-11. The Hit-and-Miss Manufacturing Company produces items that have a probability \( p \) of being defective. These items are produced in lots of 150. Past experience indicates that \( p \) for an entire lot is either 0.05 or 0.25. Furthermore, in 80 percent of the lots produced, \( p \) equals 0.05 (so \( p = 0.25 \) in 20 percent of the lots). These items are then used in an assembly, and ultimately their quality is determined before the final assembly leaves the plant. Initially the company can either screen each item in a lot at a cost of $10 per item and replace defective items or use the items directly without screening. If the latter action is chosen, the cost of rework is ultimately $100 per defective item. Because screening requires scheduling of inspectors and equipment, the decision to screen or not screen must be made 2 days before the screening is to take place. However, one item can be taken from the lot and sent to a laboratory for inspection, and its quality (defective or nondefective) can be reported before the screen/no screen decision must be made. The cost of this initial inspection is $125.

(a) Develop a decision analysis formulation of this problem by identifying the decision alternatives, the states of nature, and the payoff table if the single item is not inspected in advance.
16.5-2. A transition matrix \( P \) is said to be doubly stochastic if the sum over each column equals 1; that is,

\[
\sum_{\ell=0}^{M} p_{\ell j} = 1, \quad \text{for all} \ j.
\]

If such a chain is irreducible, aperiodic, and consists of \( M + 1 \) states, show that

\[
\pi_j = \frac{1}{M + 1}, \quad \text{for} \ j = 0', 1, \ldots, M.
\]

16.5-3. Reconsider Prob. 16.3-3. Use the results given in Prob. 16.5-2 to find the steady-state probabilities for this Markov chain. Then find what happens to these steady-state probabilities if, at each step, the probability of moving one point clockwise changes to 0.9 and the probability of moving one point counterclockwise changes to 0.1.

16.5-4. The leading brewery on the West Coast (labeled \( A \)) has hired an OR analyst to analyze its market position. It is particularly concerned about its major competitor (labeled \( B \)). The analyst believes that brand switching can be modeled as a Markov chain using three states, with states \( A \) and \( B \) representing customers drinking beer produced from the aforementioned breweries and state \( C \) representing all other brands. Data are taken monthly, and the analyst has constructed the following (one-step) transition matrix from past data.

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>0.7</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>( B )</td>
<td>0.2</td>
<td>0.75</td>
<td>0.05</td>
</tr>
<tr>
<td>( C )</td>
<td>0.1</td>
<td>0.1</td>
<td>0.8</td>
</tr>
</tbody>
</table>

What are the steady-state market shares for the two major breweries?

16.5-5. Consider the following blood inventory problem facing a hospital. There is need for a rare blood type, namely, type AB, Rh negative blood. The demand \( D \) (in pints) over any 3-day period is given by

\[
P(D = 0) = 0.4, \quad P(D = 1) = 0.3, \quad P(D = 2) = 0.2, \quad \text{and} \quad P(D = 3) = 0.1.
\]

Note that the expected demand is 1 pint, since \( E(D) = 0.3(1) + 0.2(2) + 0.1(3) = 1 \). Suppose that there are 3 days between deliveries. The hospital proposes a policy of receiving 1 pint at each delivery and using the oldest blood first. If more blood is required than is on hand, an expensive emergency delivery is made. Blood is discarded if it is still on the shelf after 21 days. Denote the state of the system as the number of pints on hand just after a delivery. Thus, because of the discarding policy, the largest possible state is 7.

(a) Construct the (one-step) transition matrix for this Markov chain.

(b) Find the steady-state probabilities of the state of the Markov chain.

(c) Use the results from part (b) to find the steady-state probability that a pint of blood will need to be discarded during a 3-day period. (Hint: Because the oldest blood is used first, a pint reaches 21 days only if the state was 7 and then \( D = 0 \)).

(d) Use the results from part (b) to find the steady-state probability that an emergency delivery will be needed during the 3-day period between regular deliveries.

16.5-6. In the last subsection of Sec. 16.5, the (long-run) expected average cost per week (based on just ordering costs and unsatisfied demand costs) is calculated for the inventory example of Sec. 16.1. Suppose now that the ordering policy is changed to the following. Whenever the number of cameras on hand at the end of the week is 0 or 1, an order is placed that will bring this number up to 3. Otherwise, no order is placed.

Recalculate the (long-run) expected average cost per week under this new inventory policy.

16.5-7. Consider the inventory example introduced in Sec. 16.1, but with the following change in the ordering policy. If the number of cameras on hand at the end of each week is 0 or 1, two additional cameras will be ordered. Otherwise, no ordering will take place. Assume that the storage costs are the same as given in the second subsection of Sec. 16.5.

(a) Find the steady-state probabilities of the state of this Markov chain.

(b) Find the long-run expected average storage cost per week.

16.5-8. Consider the following inventory policy for a certain product. If the demand during a period exceeds the number of items available, this unsatisfied demand is backlogged; i.e., it is filled when the next order is received. Let \( Z_n \) \( (n = 0, 1, \ldots) \) denote the amount of inventory on hand minus the number of units backlogged before ordering at the end of period \( n \) \( (Z_0 = 0) \). If \( Z_n \) is zero or positive, no orders are backlogged. If \( Z_n \) is negative, then \( -Z_n \) represents the number of backlogged units and no inventory is on hand. At the end of period \( n \), if \( Z_n < 1 \), an order is placed for \( 2m \) units, where \( m \) is the smallest integer such that \( Z_n + 2m \geq 1 \). Orders are filled immediately.

Let \( D_1, D_2, \ldots \) be the demand for a product in periods 1, 2, \ldots, respectively. Assume that the \( D_n \) are independent and identically distributed random variables taking on the values 0, 1, 2, 3, 4, each with probability \( \frac{1}{4} \). Let \( X_n \) denote the amount of stock on hand after ordering at the end of period \( n \) (where \( X_0 = 0 \)), so that

\[
X_n = \begin{cases} 
X_{n-1} - D_{n} + 2m & \text{if} \ X_{n-1} - D_{n} < 1 \\
X_{n-1} & \text{if} \ X_{n-1} - D_{n} \geq 1 
\end{cases} \quad (n = 1, 2, \ldots)
\]

when \( \{X_n\} \ (n = 0, 1, \ldots) \) is a Markov chain. It has only two states, 1 and 2, because the only time that ordering will take place is when \( Z_n = 0, -1, -2, \text{ or } -3\), in which case 2, 2, 4, and 4 units are ordered, respectively, leaving \( X_n = 2, 1, 2, 1 \), respectively.

(a) Construct the (one-step) transition matrix.

(b) Use the steady-state equations to solve manually for the steady-state probabilities.

(c) Now use the result given in Prob. 16.5-2 to find the steady-state probabilities.

(d) Suppose that the ordering cost is given by \( (2 + 2m) \) if an order is placed and zero otherwise. The holding cost per period is \( Z_n \) if \( Z_n \geq 0 \) and zero otherwise. The shortage cost per
16.5-9. An important unit consists of two components placed in parallel. The unit performs satisfactorily if one of the two components is operating. Therefore, only one component is operated at a time, but both components are kept operational (capable of being operated) as often as possible by repairing them as needed. An operating component breaks down in a given period with probability 0.2. When this occurs, the parallel component takes over, if it is operational, at the beginning of the next period. Only one component can be repaired at a time. A repair of a component starts at the beginning of the first available period and is completed at the end of the next period. Let \( X_t \) be a vector consisting of two elements \( U_t \) and \( V_t \), where \( U_t \) represents the number of components that are operational at the end of period \( t \) and \( V_t \) represents the number of components of repair that have been completed on components that are not yet operational. Thus, \( V_t = 0 \) if \( U_t = 2 \) or if \( U_t = 1 \) and the repair of the nonoperational component is just getting underway. Because a repair takes two periods, \( V_t = 1 \) if \( U_t = 0 \) (since then one nonoperational component is waiting to begin repair while the other one is entering its second period of repair) or if \( U_t = 1 \) and the nonoperational component is entering its second period of repair. Therefore, the state space consists of the four states: \((2,0),(1,0),(0,1),\) and \((1,1)\). Denote these four states by \( 0,1,2,3 \), respectively. \( X_t \) (\( t = 0,1,2,3 \)) is a Markov chain (assume that \( X_0 = 0 \)) with the (one-step) transition matrix

\[
P = \begin{bmatrix}
0 & 0.8 & 0 & 0 \\
1 & 0 & 0 & 0.2 \\
2 & 0 & 1 & 0 \\
3 & 0.8 & 0.2 & 0
\end{bmatrix}
\]

(a) What is the probability that the unit will be inoperable (because both components are down) after \( n \) periods, for \( n = 2,5,10,20? \)

(b) What are the steady-state probabilities of the state of this Markov chain?

(c) If it costs $30,000 per period when the unit is inoperable (both components down) and zero otherwise, what is the (long-run) expected average cost per period?

16.6.1. A computer is inspected at the end of every hour. It is found to be either working (up) or failed (down). If the computer is found to be up, the probability of its remaining up for the next hour is 0.90. If it is down, the computer is repaired, which may require more than one hour. Whenever the computer is down (regardless of how long it has been down), the probability of its still being down 1 hour later is 0.35.

(a) Construct the (one-step) transition matrix.

(b) Use the approach described in Sec. 16.6 to find the \( \mu_e \) (the expected first passage time from state \( i \) to state \( j \)) for all \( i \) and \( j \).

16.6.2. A manufacturer has a machine that, when operational at the beginning of a day, has a probability of 0.1 of breaking down sometime during the day. When this happens, the repair is done the next day and completed at the end of that day. (a) Formulate the evolution of the status of the machine as a Markov chain by identifying three possible states at the end of each day, and then constructing the (one-step) transition matrix.

(b) Use the approach described in Sec. 16.6 to find the \( \mu_0 \) (the expected first passage time from state \( i \) to state \( j \)) for all \( i \) and \( j \). Use these results to identify the expected number of full days that the machine will remain operational before the next breakdown after a repair is completed.

(c) Now suppose that the machine already has gone 20 full days without a breakdown since the last repair was completed. How does the expected number of full days hereafter that the machine will remain operational before the next breakdown compare with the corresponding result from part (b) when the repair had just been completed? Explain.

16.6.3. Reconsider Prob. 16.6.2. Now suppose that the manufacturer keeps a spare machine that only is used when the primary machine is being repaired. During a repair day, the spare machine has a probability of 0.1 of breaking down, in which case it is repaired the next day. Denote the state of the system by \((x,y)\), where \( x \) and \( y \), respectively, take on the values 1 or 0 depending upon whether the primary machine \( x \) and the spare machine \( y \) are operational (value of 1) or not operational (value of 0) at the end of the day. (Hint: Note that \( 0,0 \) is not a possible state.)

(a) Construct the (one-step) transition matrix for this Markov chain.

(b) Find the expected recurrence time for the state \((1,0)\).

16.6.4. Consider the inventory example presented in Sec. 16.1 except that demand now has the following probability distribution:

\[
P[D = 0] = \frac{1}{4}, \quad P[D = 1] = \frac{1}{2}, \quad P[D \geq 3] = 0.
\]

The ordering policy now is changed to ordering just 2 cameras at the end of the week if none are in stock. As before, no order is placed if there are any cameras in stock. Assume that there is one camera in stock at the time (the end of a week) the policy is instituted.

(a) Construct the (one-step) transition matrix.

(b) Find the probability distribution of the state of this Markov chain \( n \) weeks after the new inventory policy is instituted, for \( n = 2,5,10 \).

(c) Find the \( \mu_{e} \) (the expected first passage time from state \( i \) to state \( j \)) for all \( i \) and \( j \).

(d) Find the steady-state probabilities of the state of this Markov chain.

(e) Assuming that the store pays a storage cost for each camera remaining on the shelf at the end of the week according to the function \( C(D) = 0, C(1) = 2, \) and \( C(2) = 8 \), find the long-run expected average storage cost per week.

16.6.5. A production process contains a machine that deteriorates rapidly in both quality and output under heavy usage, so that it is
inspected at the end of each day. Immediately after inspection, the condition of the machine is noted and classified into one of four possible states:

<table>
<thead>
<tr>
<th>State</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Good as new</td>
</tr>
<tr>
<td>1</td>
<td>Operable—minimum deterioration</td>
</tr>
<tr>
<td>2</td>
<td>Operable—major deterioration</td>
</tr>
<tr>
<td>3</td>
<td>Inoperable and replaced by a good-as-new machine</td>
</tr>
</tbody>
</table>

The process can be modeled as a Markov chain with its (one-step) transition matrix \( P \) given by

<table>
<thead>
<tr>
<th>State</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>7</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Find the steady-state probabilities.
(b) If the costs of being in states 0, 1, 2, 3, are 0, $1,000, $3,000, and $6,000, respectively, what is the long-run expected average cost per day?
(c) Find the expected recurrence time for state 0 (i.e., the expected length of time a machine can be used before it must be replaced).

16.7-1. Consider the following gambler’s ruin problem. A gambler bets $1 on each play of a game. Each time, he has a probability \( p \) of winning and probability \( q = 1 - p \) of losing the dollar bet. He will continue to play until he goes broke or nets a fortune of \( T \) dollars. Let \( X_n \) denote the number of dollars possessed by the gambler after the \( n \)th play of the game. Then

\[
\begin{align*}
X_{n+1} &= X_n + 1 \quad \text{with probability } p \\
X_{n+1} &= X_n - 1 \quad \text{with probability } q = 1 - p \\
X_{n+1} &= X_n \quad \text{for } X_n = 0, \text{ or } T.
\end{align*}
\]

\( \{X_n\} \) is a Markov chain. The gambler starts with \( X_0 \) dollars, where \( X_0 \) is a positive integer less than \( T \).
(a) Construct the (one-step) transition matrix of the Markov chain.
(b) Find the classes of the Markov chain.
(c) Let \( T = 3 \) and \( p = 0.3 \). Using the notation of Sec. 16.7, find \( f_{10}, f_{1e}, f_{20}, f_{3e} \).
(d) Let \( T = 3 \) and \( p = 0.7 \). Find \( f_{10}, f_{1e}, f_{20}, f_{3e} \).

16.7-2. A video cassette recorder manufacturer is so certain of its quality control that it is offering a complete replacement warranty if a recorder fails within 2 years. Based upon compiled data, the company has noted that only 1 percent of its recorders fail during the first year, whereas 5 percent of the recorders that survive the first year will fail during the second year. The warranty does not cover replacement recorders.
(a) Formulate the evolution of the status of a recorder as a Markov chain whose states include two absorption states that involve needing to honor the warranty or having the recorder survive the warranty period. Then construct the (one-step) transition matrix.
(b) Use the approach described in Sec. 16.7 to find the probability that the manufacturer will have to honor the warranty.

16.8-1. Reconsider the example presented at the end of Sec. 16.8. Suppose now that a third machine, identical to the first two, has been added to the shop. The one maintenance person still must maintain all the machines.
(a) Develop the rate diagram for this Markov chain.
(b) Construct the steady-state equations.
(c) Solve these equations for the steady-state probabilities.

16.8-2. The state of a particular continuous time Markov chain is defined as the number of jobs currently at a certain work center, where a maximum of three jobs are allowed. Jobs arrive individually. Whenever fewer than three jobs are present, the time until the next arrival has an exponential distribution with a mean of \( \frac{1}{4} \) day. Jobs are processed at the work center one at a time and then leave immediately. Processing times have an exponential distribution with a mean of \( \frac{1}{3} \) day.
(a) Construct the rate diagram for this Markov chain.
(b) Write the steady-state equations.
(c) Solve these equations for the steady-state probabilities.