1 Simplex Method

1.1 Outline
Linear Programming so far
- Standard Form
- Equality Constraints
- Solutions, Extreme Points, and Bases
- The Representation Theorem
- Proof of the Optimal Basis Theorem

The Simplex Method
- Simplex Method: Basic Change
- Converting the Standard Form to the Augmented Form
- Simplex Example
- Getting a Basic Feasible Solution
- Phase 1 Approach
- Big M method
- Determining Optimality of a Basis

1.2 Linear Programming so far
Up till now in class we had covered the following aspects of linear programming:
- Formulate linear programs in various contexts.
- Transforming linear programs into standard form.
- The intuition behind the Simplex Method – to find the best corner point feasible solution.
- Some of the mathematical details:
  - Corner point feasible or basic feasible solutions correspond to a set of $n$ active constraints.
  - Any set of active constraints corresponds to a basis from the matrix $A$.
  - The basis is a set of linearly independent columns.
1.3 Standard Form

Standard form for a linear programming problem is:

Maximize $c_1x_1 + c_2x_2 + \ldots + c_Nx_N$

subject to

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N \leq b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N \leq b_2$
\[\ldots\]
$a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N \leq b_M$
$x_j \geq 0, j = 1..N$

The concise version is:

Maximize $c'x$

\[s.t. Ax \leq b\]
\[x \geq 0\]

where $A$ is an $m$ by $n$ matrix: $n$ variables, $m$ constraints.

1.4 Equality Constraints

In cases where the problem contains an equality constraint, it can be replaced by two inequality constraints. For example:

$3x_1 + x_2 - 3x_2 = 10$

can be replaced by

$3x_1 + x_2 - 3x_2 \leq 10$

and

$3x_1 + x_2 - 3x_2 \geq 10$

The second constraint can be negated in this case, yielding:

$-3x_1 - x_2 + 3x_2 \leq -10$

1.5 Solutions, Extreme Points, and Bases

- **Key Fact:** If a Linear Program has an optimal solution, then it has an optimal extreme point solution.

- **Basic Feasible Solution (Corner Point Feasible):** The vector $x$ is an extreme point of the solution space iff it is a BFS or $Ax = b$, $x \geq 0$.

- If $A$ is of full rank then there is at least one basis $B$ of $A$. $B$ is a set of linearly independent columns of $A$.

- $B$ gives us a basic solution. If this is feasible then it is called a basic feasible solution (BFS) or corner point feasible (CPF).
1.6 The Representation Theorem

The proof of the Optimal Basis Theorem requires the Representation Theorem.

**Representation Theorem:** Consider the set $S = \{x : Ax = b, x \geq 0\}$.

Let $V = \{V_1 \ldots V_N\}$ be the extreme points of $S$. If $S$ is nonempty, $V$ is nonempty and every feasible point $x \in S$ can be written as:

$$x = d + \sum_{i=1}^{k} \alpha_i V_i$$

where $\sum \alpha_i = 1, \alpha_i \geq 0 \forall i$

and $d$ satisfies $Ad = 0, d \geq 0$

A simple explanation of this theorem is that every point in a solution space is a combination of the convex corners of that space. We discard the unbounded cases for simplification in the following proof:

1.7 Proof of the Optimal Basis Theorem

Say $x$ is a finite optimal solution of a linear program.

The representation theorem then says:

$$x = d + \sum_{i=1}^{k} \alpha_i V_i$$

**case** $d = 0$

$$c^T x = c^T \sum_{i=1}^{k} \alpha_i V_i$$

$$= \sum_{i=1}^{k} c^T \alpha_i V_i$$

It is known that $\sum \alpha_i = 1$ and that $\alpha_i \geq 0$.

Let $V^* = \arg \max c^T V_i$

then $c^T x \leq \sum \alpha_i c^T V^* = c^T V^*$

Therefore $c^T x \leq c^T V^*$

**Optimal basis theorem** If a linear program has a finite optimal solution then it has an optimal basic feasible solution.

2 The Simplex Method

The intuition behind the Simplex method is that it checks the corner points and gets a better solution at each iteration.

A simplified outline of the Simplex method:

1. Find a starting solution.
2. Test for optimality.
   - If optimal then stop.
2.1 Simplex Method: Basis Change

1. One basic variable is replaced by another.
   - This process essentially chooses which variables are active and then changes the active constraints.

2. The optimality test identifies a non-basic variable to enter the basis.
   - The entering basic variable is increased until one of the other basic variables becomes zero.
   - The variable that reaches zero is identified using the minimum ratio test.
   - That variable departs the basis.

2.2 Converting the Standard Form to the Augmented Form

Doing this conversion allows you to apply the Simplex method to the augmented form and is relatively straightforward. The gist of the idea is to add a slack variable to add a slack variable to replace each inequality.

An example would be converting from the following left set of equations to the following right set:

\[
\begin{align*}
\text{max } & \quad c_1x_1 + c_2x_2 + \ldots + c_Nx_N \\
\text{subject to } & \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N \leq b_1 \\
& \quad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N \leq b_2 \\
& \quad \ldots \\
& \quad a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N \leq b_M \\
& \quad x_j \geq 0, j = 1..N \\
\end{align*}
\]

\[
\begin{align*}
\text{max } & \quad c_1x_1 + c_2x_2 + \ldots + c_Nx_N \\
\text{subject to } & \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N + \bar{x}_1 = b_1 \\
& \quad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N + \bar{x}_2 = b_2 \\
& \quad \ldots \\
& \quad a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N + \bar{x}_N = b_M \\
& \quad x_j, \bar{x}_j \geq 0, j = 1..N, i = 1..m \\
\end{align*}
\]

A more concise representation of the same conversion is:

\[
\begin{align*}
\text{max } & \quad c'x \\
\text{subject to } & \quad Ax \leq b \\
& \quad x \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{max } & \quad c'x \\
\text{subject to } & \quad [A, I] \begin{bmatrix} x \\ \bar{x} \end{bmatrix} = b \\
& \quad x \geq 0 \\
\end{align*}
\]

A is an \( m \) by \( n \) matrix: \( n \) variables, \( m \) constraints

2.3 Simplex Example

The following examples are from the OR tutor CD. This is one iteration of the Simplex method via the algebraic method:
To demonstrate the simplex method, consider the following linear programming model:

\[ \text{Maximize } Z = 20x_1 + 10x_2 \]

subject to

\[ x_1 - x_2 \leq 1 \]
\[ 3x_1 + x_2 \leq 7 \]

and

\[ x_1 \geq 0, \quad x_2 \geq 0. \]

This is the model for Leo Coco’s problem presented in the demo, Graphical Method. That demo describes how to find the optimal solution graphically, as displayed on the right.

Thus the optimal solution is \( x_1 = 0, \quad x_2 = 7, \) and \( Z = 70. \)

We will now describe how the simplex method (an algebraic procedure) obtains this solution algebraically.
The Simplex Formulation

To solve this model, the simplex method needs a system of equations instead of inequalities for the functional constraints. The demo, Interpretation of Slack Variables, describes how this system of equations is obtained by introducing nonnegative slack variables, $\alpha_6$ and $\alpha_4$. The resulting equivalent form of the model is

\[
\text{Maximize } Z = 20x_1 + 10x_2 \\
\text{subject to } \begin{align*}
2x_1 &- x_2 \leq 1 \\
3x_1 &+ x_2 \leq 7 \\
\text{and } &\quad x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]

The simplex method begins by focusing on equations (1) and (2) above.

\[
\begin{align*}
(0) \quad & Z - 20x_1 - 10x_2 = 0 \\
(1) \quad & x_1 - x_2 + x_6 = 1 \\
(2) \quad & 3x_1 + x_2 + x_4 = 7 \\
\text{and } &\quad x_1 \geq 0, \quad x_2 \geq 0, x_6 \geq 0, x_4 \geq 0.
\end{align*}
\]
Algebraic Simplex Method - Initial Solution

Consider the initial system of equations exhibited above. Equations (1) and (2) include two more variables than equations. Therefore, two of the variables (the nonbasic variables) can be arbitrarily assigned a value of zero in order to obtain a specific solution (the basic solution) for the other two variables (the basic variables). This basic solution will be feasible if the value of each basic variable is nonnegative. The best of the basic feasible solutions is known to be an optimal solution, so the simplex method finds a sequence of better and better basic feasible solutions until it finds the best one.

To begin the simplex method, choose the slack variables to be the basic variables, so $x_1$ and $x_2$ are the nonbasic variables to set equal to zero. The values of $x_3$ and $x_4$ now can be obtained from the system of equations.

The resulting basic feasible solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 7$. Is this solution optimal?
Algebraic Simplex Method - Checking Optimality

To test whether the solution \( x_1 = 0 \), \( x_2 = 0 \), \( x_3 = 1 \), and \( x_4 = 7 \) is optimal, we rewrite equation (0) as

\[
Z = 0 + 20x_1 + 10x_2
\]

Since both \( x_1 \) and \( x_2 \) have positive coefficients, \( Z \) can be increased by increasing either one of these variables. Therefore, the current basic feasible solution is not optimal, so we need to perform an iteration of the simplex method to obtain a better basic feasible solution.

This begins by choosing the entering basic variable (the nonbasic variable chosen to become a basic variable for the next basic feasible solution).

Algebraic Simplex Method - Entering Basic Variable

The entering basic variable is: \( x_1 \).

Why? Again, rewrite equation (0) as \( Z = 0 + 20x_1 + 10x_2 \).

The value of the entering basic variable will be increased from 0. Since \( x_1 \) has the largest positive coefficient, increasing \( x_1 \) will increase \( Z \) at the fastest rate. So select \( x_1 \).

This selection rule tends to minimize the number of iterations needed to reach an optimal solution. You'll see later that this particular problem is an exception where this rule does not minimize the number of iterations.

Selecting an Entering Basic Variable

The entering basic variable is: \( x_1 \).

Why? Again rewrite equation (0) as \( Z = 0 + 20x_1 + 10x_2 \).

The value of the entering basic variable will be increased from 0. Since \( x_1 \) has the largest positive coefficient, increasing \( x_1 \) will increase \( Z \) at the fastest rate. So select \( x_1 \).

This selection rule tends to minimize the number of iterations needed to reach an optimal solution. You'll see later that this particular problem is an exception where this rule does not minimize the number of iterations.
Algebraic Simplex Method - Leaving Basic Variable

(0) $Z - 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0$
(1) $1x_1 - 1x_2 + 1x_3 + 0x_4 = 1$
(2) $3x_1 + 1x_2 + 0x_3 + 1x_4 = 7$

Selecting a Leaving Basic Variable

The entering basic variable is: $x_1$

The leaving basic variable is: $x_2$

Why? Choose the basic variable that reaches zero first as the entering basic variable ($x_1$) is increased (watch $x_1$ increase).

$x_2 = 0$ when $x_1 = 1$.

What if we increase $x_1$ until $x_4 = 0$?
Algebraic Simplex Method - Leaving Basic Variable

(0) \[ Z - 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \]

(1) \[ 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \]

(2) \[ 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7 \]

Selecting a Leaving Basic Variable

The entering basic variable is: \( x_1 \)

The leaving basic variable is: \( x_4 \)

Why? Choose the basic variable that reaches zero first as the entering basic variable (\( x_1 \)) is increased.

\[ x_0 = 0 \text{ when } x_1 = 1. \]

What if we increase \( x_1 \) until \( x_4 = 0 \) (watch \( x_1 \) increase)?

\[ x_4 = 0 \text{ when } x_1 = 7/3. \]

However \( x_4 \) is now negative, resulting in an infeasible solution. Therefore, \( x_4 \) cannot be the leaving basic variable.

Algebraic Simplex Method - Gaussian Elimination

(0) \[ Z - 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \]

(1) \[ 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \]

(2) \[ 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7 \]

Scaling the Pivot Row

In order to determine the new basic feasible solution, we need to convert the system of equations into proper form from Gaussian elimination. The coefficient of the entering basic variable (\( x_1 \)) in the equation of the leaving basic variable (equation (1)) must be 1.

The current value of this coefficient is: 1

Therefore, nothing needs to be done to this equation.
Algebraic Simplex Method - Gaussian Elimination

Eliminating $x_1$ from the Other Equations

Next, we need to obtain a coefficient of zero for the entering basic variable ($x_1$) in every other equation (equations (0) and (2)).

The coefficient of $x_1$ in equation (0) is: -20

To obtain a coefficient of 0 we need to:

- Add 20 times equation (1) to equation (0).

The coefficient of $x_1$ in equation (2) is: 3

Therefore, to obtain a coefficient of 0 we need to:

- Subtract 3 times equation (1) from equation (2).

Algebraic Simplex Method - Checking Optimality

Checking for Optimality

The new basic feasible solution is $x_1 = 1, x_2 = 0, x_3 = 0,$ and $x_4 = 4,$ which yields $Z = 20$.

This ends iteration 1.

Is the current solution optimal? No.

Why? Rewrite equation (0) as $Z = 20 + 30x_2 - 20x_3$.

Since $x_2$ has a positive coefficient, increasing $x_2$ from zero will increase $Z$. So the current basic feasible solution is not optimal.
Tabular Simplex Method - Introduction

To demonstrate the simplex method in tabular form, consider the following linear programming model:

\[
\begin{align*}
\text{Maximize } & \quad Z = 20x_1 + 10x_2 \\
\text{subject to } & \quad x_1 - x_2 \leq 1 \\
& \quad 3x_1 + x_2 \leq 7 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]

This is the same problem used to demonstrate the simplex method in algebraic form (see the demo The Simplex Method - Algebraic Form), which yielded the optimal solution \((x_1, x_2) = (0, 7)\), as shown to the right.

Tabular Simplex Method - Initial Tableau

Using the Tableau

After introducing slack variables \((x_3, x_4)\), etc., the initial tableau is as shown above.

Choose the slack variables to be basic variables, so \(x_3\) and \(x_4\) are the nonbasic variables to be set to zero. The values of \(x_3\) and \(x_4\) can now be obtained from the right-hand side column of the simplex tableau.

The resulting basic feasible solution is \(x_1 = 0, \ x_2 = 0, \ x_3 = 1, \ \text{and } x_4 = 7.\)
Tabular Simplex Method - Entering Basic Variable

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1 -20 -10 0 0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>0 1 -1 1 0</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0 3 1 0 1</td>
<td>7</td>
</tr>
</tbody>
</table>

Selecting an Entering Basic Variable

The entering basic variable is: $x_1$

Why? This is the variable that has the largest (in absolute value) negative coefficient in row 0 (the equation (0) row).

Tabular Simplex Method - Leaving Basic Variable

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Ratio Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1 -20 -10 0 0</td>
<td>Minimum</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>0 1 -1 1 0</td>
<td>1/1 $x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0 3 1 0 1</td>
<td>7/3</td>
</tr>
</tbody>
</table>

Selecting a Leaving Basic Variable

The entering basic variable is: $x_1$

The leaving basic variable is: $x_3$

Why? Apply the minimum ratio test as shown above.
Tabular Simplex Method - Gaussian Elimination

Scaling the Pivot Row
Although it is not needed in this case, the pivot row is normally divided by the pivot number.

Eliminating \( x_1 \) from the Other Rows
Add 20 times row 1 to row 0.
Subtract 3 times row 1 from row 2.

Why? To obtain a new value of zero for the coefficient of the entering basic variable in every other row of the simplex tableau.
Tabular Simplex Method - Gaussian Elimination

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of $x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>-20</td>
<td>-10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>7</td>
</tr>
</tbody>
</table>

Eliminating $x_1$ from the Other Rows

Add 20 times row 1 to row $0$.

Subtract 3 times row 1 from row $2$.

Why? To obtain a new value of zero for the coefficient of the entering basic variable in every other row of the simplex tableau.

Tabular Simplex Method - Checking Optimality

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of $x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-30</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-5</td>
<td>1</td>
</tr>
</tbody>
</table>

Checking for Optimality

This ends iteration 1.

Is the current basic feasible solution optimal? No.

Why? Because row $0$ includes at least one negative coefficient.

2.4 Getting a Basic Feasible Solution

Given a linear program where $B$ is the basis of $A$:

$$\text{Maximize } c'x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

To get a basic solution rewrite the linear program in terms of the basis:
\[
A = [BN] \\
Ax = Bx_B + Nx_N = b
\]

\[
x_B \in \mathbb{R}^m, \quad x_N \in \mathbb{R}^{n-m} \\
x_N = 0, \quad x_B = B^{-1}b
\]

\(x\) is a basic feasible solution if it is feasible.

### 2.5 Phase 1 approach

Initially, revise the constrains of the original problem by introducing artificial variables as needed to obtain an obvious initial BF solution for the artificial problem. Phase 1: The objective for this phase is to find a BF solution for the real problem. To do this, Minimize \(Z = \text{the sum of the artificial variables, subject to revised constrains}\).

Phase 2: The objective of this phase is to find an optimal solution for the real problem. Since the artificial variables are not part of the real problem, these variables can now be dropped. Starting from the BF solution obtained at the end of phase 1, use the simplex method to solve the real problem. pg 135

\[
x : Ax = b \\
x \geq 0
\]

look for solution to 2:
\(y = 0\)
and \(y = b\) and \(x = 0\) for 1
solve phase 1 problem

Minimize \(e' y \text{feasibilities}\)

\[
s.t. Ax + Iy = b \\
x \geq 0 \\
y \geq 0
\]

if \(y = 0\)
then you have a problem because you are missing a term

Solve phase 2 \(Ax + Iy = b\)

\[
x \geq 0 \\
y \geq 0
\]
2.6 Big M method

Solve original Linear Program with objective plus large number

\[
\begin{align*}
\text{Maximize} & \quad c^'x - Me^'y \\
\text{s.t.} & \quad Ax + Iy = b \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

Computation worked out on page 131 of OR

2.7 Determining Optimality of a Basis

Once you have a basic feasible solution in the following form:

\[
\begin{align*}
\text{Maximize} & \quad c^B x_B + c^N x_N \\
\text{s.t.} & \quad Bx_B + Nx_N = B \\
& \quad x_B, x_N \geq 0
\end{align*}
\]

Algebraically manipulating the program we can solve for \(x_B\):

\[
\begin{align*}
Bx_B + Nx_N &= b \\
x_B &= B^{-1}(b - Nx_N)
\end{align*}
\]

Then rearranging \(c^'x\):

\[
\begin{align*}
c^'x &= c^B x_B + c^N x_N \\
&= c^B B^{-1}(b - Nx_N) + c^N X_N \\
&= c^B B^{-1}b - c^B B^{-1} Nx_N + c^N x_N \\
&= z_0 - \sum_j (z_j - c_j)x_j
\end{align*}
\]

where:

\[
\begin{align*}
z_0 &= c^B B^{-1}b \\
z_j &= c^B B^{-1}A_j \quad j \in N
\end{align*}
\]

We can now see that:

\[
\max c^'x = z_0 - \sum_j (z_j - c_j)x_j
\]

So to test optimality we can check:

\[
z_j - c_j \geq 0 \forall j \in N
\]