1 Introduction

Optimization often involves considering the behavior of others, as a result, we need to take this into account when looking for optimal solutions. This is where game theory comes in. Game theory is a mathematical theory that deals with the general features of competitive situations in a formal, abstract way. It places particular emphasis on the decision making processes of the adversaries. Typically, an optimal course of action will involve input from our competitors and/or collaborators. Standard assumption used during our treatment of game theory:

- Everyone involved is greedy.
- Everyone trying to maximize/minimize her/his own objectives.
- The outcome of the “agent” or “player” depends on the decision of other agents.
- Everyone involved is rational.

2 Simplest game:

Generally two-person, zero-sum games are considered the simplest games. One player’s gain is another player’s loss. In general a two-person game is characterized by

- The strategies of player 1
- The strategies of player 2
- The payoff table

3 Dominated Strategy

“Dominated Strategy:” Another strategy is at least as good for every opponent’s strategy. This essentially means that if there is a move that obviously will not benefit a player, then that move is said to be dominated and taken out of consideration.

Example: Political Strategy

In this example two political candidates are competing for a same position. Each candidate will be campaigning in cities A and B for two days, and they want to maximize the number of votes they gain by spending their time in the cities accordingly. They have to decide on what combination of two cities to visit in final two days of campaign. Possible scenarios:

- Stay in city A for both days.
- Go to city A one day and city B the next or viceversa.
Table 1: Payoff table for player 1 used in example 1.

<table>
<thead>
<tr>
<th></th>
<th>P2 1</th>
<th>P2 2</th>
<th>P2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 1</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>P1 2</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>P1 3</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 2: Payoff table used in example 2.

<table>
<thead>
<tr>
<th></th>
<th>P2 1</th>
<th>P2 2</th>
<th>P2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 1</td>
<td>-3</td>
<td>-2</td>
<td>6</td>
</tr>
<tr>
<td>P1 2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>P1 3</td>
<td>5</td>
<td>-2</td>
<td>-4</td>
</tr>
</tbody>
</table>

- Stay in city B for both days.

The benefits of each option is shown in the payoff table in Table ???. It is never a good idea for either player to do strategy 3, so those strategies are said to be “dominated.”

We can continue analyzing the payoff table to find another dominated strategy: strategy 2 for player 1. This leaves one choice left for player 1 and two choices for player 2. Player 2 is rational and therefore will choose the strategy with the lowest payoff for player 1, strategy 1. This gives player 1 a final payoff of one. The fact that there are enough dominated strategies to know that one player will win over the other implies that this is a “unfair game.” A “fair game” would be one where the final payoff for either player would be zero.

4 Another table:

Take a look at Table ???. In this table there is no dominated row or column. Therefore no dominated strategy can be figured out. This scenario leads to a concept of risk aversion.

5 Risk Aversion

“Risk Aversion:” People prefer to minimize the worst case move rather than maximize the best possible case. This generalization is used to determine rational behavior of players. For example in Table ???, worst case for player one is -4. Therefore rule out row 3. Next possible worst case could be -3 and so on. Worst case for player (P2) is the gain for player (P1). Therefore worst case for player (P2) is column 1 and 3 in Table ???.

6 Mixed Strategy

6.1 Unstable Problem

If the assumption of rational players leads to a stalemate, then the problem/game is said to be “unstable.” In an unstable problem, each player could improve if they know the other player’s
strategies. The solution to this problem involves making a random decision. This is called a “mixed strategy.” Some terms:

- $x_i$ - probability P1 uses strategy $i$, $i = 1 \ldots m$
- $y_j$ - probability P2 uses strategy $j$, $j = 1 \ldots n$
- “Pure strategy:” when $x_i, y_j \in 0, 1$
- Expected payoff for P1 = $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_i y_j$

6.2 Minimax Theorem

Basically it is the strategy where we minimize the maximum expected value of loss. If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax theorem provides a stable solution with $v = v_0 = v^* = \text{value of the game}$. In other words, neither player can do better by unilaterally changing strategies. In other words it corresponds to primal/dual decisions.

6.3 Calculating the Optimal Mixed Strategy

There are two methods used to find the optimal mixed strategy for a problem:

- Graphically
- Using linear programming.

The graphical method is very limiting and only works when one of the players only has 2 undominated strategies.

6.4 Graphical Method

Given the payoff table in Table 3, we can get the 3 possible payoffs for player 2:

- $0x_1 + 5(1 - x_1) = -5x_1 + 5$
- $-2x_1 + 4 - x_1 = -6x_1 + 4$
- $2x_1 - 3 + 3x_1 = 5x_1 - 3$. 

Table 3: Payoff table for player 2 used in example 3.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>$1 - x_1$</td>
<td>5</td>
<td>4</td>
<td>-3</td>
</tr>
</tbody>
</table>
Figure ?? shows the graph of these possible strategies. For this particular problem, player 2 will pick the lowest intersection of any two strategies. This is where $-3 + 5x_1$ and $4 - 6x_1$ meet, which yields a $x_1^* = \frac{7}{11}$. So what should player 2 do?

\[
\text{Payoff} = \frac{20}{11}(y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^*) = \frac{2}{11}
\]

\[
y_1^* + y_2^* + y_3^* = 1, y_j \geq 0
\]

Note: We must have $y_1^* = 0$ or we will have a max point above \(\frac{2}{11}\). With this in mind, we can solve for $y_2^*, y_3^*$.

\[
y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \Rightarrow \begin{cases} 
\leq \frac{2}{11} & \text{for } 0 \leq x_1 \leq 1 \\
= \frac{2}{11} & \text{for } x_1 = \frac{7}{11}
\end{cases}
\]

Must have $y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) = \frac{2}{11}$ for all $x \in [0, 1]$. This gives two equations which can be solved to give $y^* = (0, \frac{5}{11}, \frac{6}{11})$. The graphical method can be helpful in understanding a solution to a particular problem, but is heavily limited in the dimensions of the problems it can handle.

### 6.5 Using Linear Programming

In general any game with mixed strategies can be solved by transforming the problem to a linear programming problem. Using the minimax theorem, calculating the optimal mixed strategy can be done using linear programming.

\[
E(\text{Payoff}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_iy_j
\]

$x_1...x_m$ are optimal if $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_iy_j \geq v = v$. This inequality must hold for all of the pure strategies of P2, i.e. whenever $y_j = 1$, rest 0:

\[
\sum_{i=1}^{n} p_{ij}x_i \geq v, j = 1...n.
\]

These cover the original since $\sum y_i = 1$,

\[
\sum_{j=1}^{n} y_j \sum_{i=1}^{m} p_{ij}x_i \geq \sum_{j=1}^{n} y_jv = v.
\]

These $n$ constraints are sufficient to original. Add $x_1 + x_2 + ... + x_m = 0, x \geq 0$, we have a set of linear constraints in $x$. Therefore, optimal mixed strategy $\leftrightarrow$ feasible linear program.

There are two problems with this approach:

1. $v$ unknown.
2. no objective

For player 1, let \( x_{m+1} = v \),

\[
\begin{align*}
\text{Max } x_{m+1} \\
\text{s.t. } & p_{11}x_1 + p_{21}x_2 + ... + p_{m1}x_m - x_{m+1} \geq 0 \\
& p_{12}x_1 + p_{22}x_2 + ... + p_{m2}x_m - x_{m+1} \geq 0 \\
& \vdots \\
& p_{1n}x_1 + p_{2n}x_2 + ... + p_{mn}x_m - x_{m+1} \geq 0 \\
\end{align*}
\]

\[x_1 + ... + x_m = 1, x_i \geq 0, i = 1, ..., m.\]

Note: \( x_{m+1} \) could be negative. A similar problem exists for player 2:

\[
\begin{align*}
\text{Max } y_{n+1} \\
\text{s.t. } & p_{11}y_1 + p_{21}y_2 + ... + p_{n1}y_n - y_{n+1} \geq 0 \\
& p_{12}y_1 + p_{22}y_2 + ... + p_{n2}y_n - y_{n+1} \geq 0 \\
& \vdots \\
& p_{1n}y_1 + p_{2n}y_2 + ... + p_{mn}y_n - y_{n+1} \geq 0 \\
\end{align*}
\]

\[y_1 + ... + y_n = 1, y_i \geq 0, i = 1, ..., n.\]

The problems for player 1 and player 2 form a dual/primal relationship. The optimal minimax strategy for both players is found by solving one linear program. This could be a great advantage if one player’s problem is significantly more complex than the other’s.

7 Extensions in Game Theory

- Two person constant sum game.
- N-person games- a more complicated extension.
- Cooperative games.
- Communication.
- Repeated/sequential games.

8 References:

- Class notes
- Previous year’s lecture
- Textbook