Indirect Imaging:
Image Reconstruction From Projections

Direct vs. Computed (Indirect) Imaging

• Direct Imaging:
  – Optical Imaging such as photography, video

• Indirect Imaging:
  – Reflection Imaging
  – Emission Imaging
  – Transmission Imaging

TOMOGRAPHY: Imaging from Slices (TOMO-)
Two Important Applications

- **Medicine**
  - CAT-Scan (Transmission)
  - MRI, fMRI (Transmission)
  - P.E.T. (Emission)
  - Ultrasound (Reflection)

- **Non-destructive Testing and Evaluation**
  - Security inspection
  - Buildings, Airplanes, etc structural inspection

- **Other Applications: Oceanography, Astronomy**
  - Borehole Tomography
  - Ocean Acoustic Tomography

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Borehole Geophysical Tomography
X-Ray Attenuation and Tomography

\[ I_{\text{meas}} = I_{\text{trans}} \exp \left(- \int_{\text{Line}} f(x, y) \, du \right) \]

Transmitted Beam Intensity

Measured Intensity Profile

\[ g(s, \theta) = \ln \left( \frac{I_{\text{trans}}}{I_{\text{meas}}} \right) \]

Ref: Kalendar, 2000

The CT Scanner

Abdominal CT
Parallel-Beam vs. Fan-Beam Data

- We will concentrate on algorithms for Parallel-Beam data
- The data can be transformed from parallel to fan-beam to parallel-beam and vice-versa

Theory and Practice of CT

- The theory behind reconstructing a function from its projections was invented by Johann Radon in 1917!
- It was not realized that this could lead to an imaging device until the mid 1960s.
- Nobel Prize in Medicine in 1979

G.N. Hounsfield  Allan Cormack
Tomography: Imaging From Projections

\[ g(s, \theta) = \iint f(x, y) \delta(x \cos \theta + y \sin \theta - s) \, dx \, dy \]

- Projection
- Raysum
- Line Integral
- Other names

Inverse Problem

Given \( g(s, \theta) \), find \( f(x, y) \).

- Incomplete, sparse data
- Noise sensitivity
- Computational complexity
Radon Transform Properties

- Linearity
  \[ af_1(x, y) + bf_2(x, y) \leftrightarrow ag_1(s, \theta) + bg_2(s, \theta) \]

- Symmetry
  \[ g(s, \theta) = g(-s, \theta \pm \pi) \]

- Periodicity
  \[ g(s, \theta) = g(s, \theta + 2k\pi) \]

- Shift
  \[ f(x - x_0, y - y_0) \leftrightarrow g(s - x_0 \cos \theta - y_0 \sin \theta, \theta) \]

One More Radon Transform Property

- Inner-Product Slice Theorem*
  \[ \int g(s, \theta)H(s)ds = \iint f(x, y)H(x \cos \theta + y \sin \theta)dxdy \]

- Proof: Plug in \[ g(s, \theta) = \iint f(x, y)\delta(x \cos \theta + y \sin \theta - s)dxdy \]

  \[ \begin{align*}
  \int g(s, \theta)H(s)ds &= \int \left[ \iint f(x, y)\delta(x \cos \theta + y \sin \theta - s)dxdy \right]H(s)ds \\
  &= \iint \left[ \delta(x \cos \theta + y \sin \theta - s)H(s)ds \right] f(x, y)dxdy \\
  &= \iint f(x, y)H(x \cos \theta + y \sin \theta)dxdy
  \end{align*} \]

*H(s) must be square-integrable
Famous Special Case:

- **Projection-Slice Theorem:** Let \( H(s) = e^{-2\pi j s \omega} \)

\[
\int g(s, \theta) H(s) ds = \iint f(x, y) H(x \cos \theta + y \sin \theta) dx dy
\]

\[
\int g(s, \theta) e^{-2\pi j s \omega} ds = \iint f(x, y) e^{-2\pi j (x \cos \theta + y \sin \theta) \omega} dx dy
\]

**Fourier Domain Reconstruction**

1. Collect many projections at diverse angles over the range 0-180 degrees. This yields many central slices in the spectrum.
2. Interpolate the given spectral samples from polar to rectangular grid.
3. Compute inverse discrete Fourier Transform to obtain an estimate of \( f(x, y) \).
Difficulties of Fourier Recons. Method

- Polar Samples
- Cartesian grid

**Problem:** Interpolation is very problematic unless there are very many lines sampled. Sampling is dense near the origin (DC) and very sparse at higher frequencies.

The Filtered Back-Projection Algorithm

- A very popular algorithm for reconstruction, FBP is based on the inversion of the Radon Transform as a linear operator.

\[ g(s, \theta) = R[f(x, y)] \]

\[ \hat{f}(x, y) = (R^* R)^{-1} R^*[g(s, \theta)] \]

*Adjoint Operator: Backprojection*
The Backprojection Operator

\[ b(x, y) = \int_{0}^{\pi} g(x \cos \theta + y \sin \theta, \theta) \, d\theta \]

- Smears each projection value evenly across the image along the corresponding line \( s = x \cos \theta + y \sin \theta \)

\[ g(s, \theta_0) \]

4 Projections

8 Projections

15 Projections

60 Projections
What is the Backprojected Image?

• A blurred version of the original image:
  \[ b(x, y) = f(x, y) * h(x, y) \]

• The blurring function is very specific:
  \[ h(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \]

• In the frequency domain:
  \[ B(\omega_x, \omega_y) = F(\omega_x, \omega_y)H(\omega_x, \omega_y) \quad \text{with} \quad H(\omega_x, \omega_y) = \frac{1}{\sqrt{\omega_x^2 + \omega_y^2}} \]

\[ F(\omega_x, \omega_y) = \sqrt{\omega_x^2 + \omega_y^2} B(\omega_x, \omega_y) \]

High-Pass (Differentiator) Filter

“Stabilized” FBP

• The differentiator (inverse) filter will amplify high frequencies, including noise and artifacts.
Where does filter come from?

- Recall the polar to cartesian grid interpolation.
- The Jacobian of the coordinate transformation gives $h(x,y)$

\[
h(x, y) = \frac{1}{\sqrt{x^2 + y^2}}
\]

\[
H(\omega_x, \omega_y) = \frac{1}{\sqrt{\omega_x^2 + \omega_y^2}}
\]

The FBP Algorithm

1. Measure projections
2. Backproject
3. Filter the result with a 2-D “differentiator”

Because of the linearity of all the steps, the following is an equivalent way:

1. Measure projections
2. Filter projections with a 1-D “differentiator”
3. Backproject
Algebraic Formulation

- Matrix formulation

\[ g(s_k, \theta) = \sum_{i=1}^{LN} \alpha_i(s_k, \theta) f_i \]

Like a convolution, but along a line

Area = \(a_1\)

Relative Area = \(\frac{a}{d^2} = \alpha_i\)

\[ d \]

\[ g(s_k, \theta) \]

\[ s_k \]

Algebraic Reconstruction Methods

- Example:

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ \tilde{g} = Af + n \]

\[ \hat{f} = (A^T A + \lambda I)^{-1} A^T \tilde{g} \]

- Matrix A is banded with bandwidth \(\text{Int}[\sqrt{2}N]\)
- Total number of nonzero elements in A is \(\text{Int}[\sqrt{2}N] \times \text{Number of projection samples}\)
- In general, A is sparse, structured \(\rightarrow\) Fast methods
A Note on Iterative Methods

- In contrast to earlier methods, here we have to resort to iterative methods involving large matrices.

- This is a huge topic called Numerical Analysis (undergraduate course Engr 147)

- As a preview, let's have a look at a basic iterative method (Richardson's Iteration) that is well-known, but does not take account of special structure.

Simple Iterative Solution for Square System

- General Problem: \( g = Af \)
- Richardson's Method: \( f_{k+1} = f_k + c_k \) \( \text{“Correction”} \)

\[
\begin{align*}
f_{k+1} &= f_k + (g - Af_k) \\
f_{k+1} &= (I - A)f_k + g
\end{align*}
\]

The residual as a good candidate for the correction term in the iteration.

Initialize iteration with the data itself: \( f_0 = g \)

\[
\begin{align*}
f_{k+1} &= (I - A)^k f_0 + (I - A)^{k-1} g + \cdots + (I - A) g + g \\
f_{k+1} &= \left[ \sum_{j=0}^{k} (I - A)^j \right] g
\end{align*}
\]

Finite number of iterations gives a regularized solution

If \( 0 < \lambda (I - A) < 1 \), then

\[ f_\infty \rightarrow A^{-1} g \]
Conclusions

• Inverse methods in imaging are a very important area of research, with many applications

• We presented methods for parallel-beam transmission tomography image reconstruction
  – Fourier Reconstruction
  – Filtered Backprojection
  – Algebraic Reconstruction

• Numerous application areas require much more in depth analysis and mathematics.