Image Restoration: The Problem and Basic Approaches

What is Restoration (vs. Enhancement):

**Enhancement**
- Making pleasing images
- Often no specific model of the degradation
- Ad hoc procedures

**Restoration**
- Undoing (inverting) and unwanted effect
- Model-based approach
- Optimality criteria
The Image Restoration Problem:

\[ g(x, y) = f(x, y) * h(x, y) + n(x, y) \]

**PROBLEM:**

Given the degraded image \( g(x, y) \) find the ideal image \( f(x, y) \)

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The Forward Model

Original image - \( f \)

\[
\begin{bmatrix}
0.0144 & 0.0281 & 0.0351 & 0.0281 & 0.0144 \\
0.0281 & 0.0547 & 0.0683 & 0.0547 & 0.0281 \\
0.0351 & 0.0683 & 0.0853 & 0.0683 & 0.0351 \\
0.0281 & 0.0547 & 0.0683 & 0.0547 & 0.0281 \\
0.0144 & 0.0281 & 0.0351 & 0.0281 & 0.0144
\end{bmatrix}
\]

Blur - \( h \)

Measured image - \( g \)

Noise - \( n \)

\( f, g, n \) of size \( M \times M \)
The Forward Model: Matrix Formulation

\[
\text{CS} = \mathbf{H} \ (\text{An } M^2 \text{ by } M^2 \text{ matrix})
\]

\[
\text{CS} = \mathbf{f} \ (\text{An } M^2 \text{ by } 1 \text{ vector})
\]

\[
\text{CS} = \mathbf{n}
\]

\[
\text{CS} = \mathbf{g}
\]

\[
\mathbf{g} = \mathbf{Hf} + \mathbf{n}
\]

A section through the image
The importance of the problem

- The image restoration problem is common in imaging systems.
- Other important problems can be solved using the same tools developed here.
- The problem and its solutions form a very interesting mathematical field, called “Inverse Problems”.

Sources of Noise

Three major sources:
- During acquisition (often random)
- During transmission
- Effects of coding/decoding

Image noise as a random variable:
- For every (x,y), n(x,y) can be considered as a random variable.
- In general, we assume that the noise n(x,y) is not dependent on the underlying signal f(x,y).
- In general, we assume that the value of n(x,y) is not correlated with n(x',y'). (Spatially uncorrelated noise)
Examples of Noise

- Gaussian
- Rayleigh
- Uniform
- Exponential

Graphs showing the probability density functions (PDFs) of different noise distributions and corresponding images.
(Easy) Examples of Degradation: Periodic Noise

Really enhancement rather than restoration

Examples of Degradation: Atmospheric Turbulence

\[ H(\omega_x, \omega_y) = e^{-k(\omega_x^2 + \omega_y^2)^{3/6}} \]

Example:

Look out the window of an airplane, behind the engine, at the ground.
Examples of Degradation: Motion blur

Example:

Photography with slow shutter speed in the presence of relative motion between the camera and the scene being imaged.

\[
h(x, y) \approx \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}
\]

**Linear Image Restoration:**

**Problem:**

\[ f(x, y) \rightarrow h(x, y) \rightarrow g_c(x, y) \oplus n(x, y) \rightarrow g(x, y) \]

Ideal image

"Blur"

Measured image

"Noise"

**Solution:**

\[ g(x, y) \rightarrow h_r(x, y) \rightarrow \hat{f}(x, y) \]

Measured image

"Restoration filter"

"Restored/Reconstructed Image"

\[ \hat{f}(x, y) = g(x, y) * h_r(x, y) \]
Image Restoration Is a Hard Problem:

\[ f(x, y) \xrightarrow{h(x, y)} g_c(x, y) \xrightarrow{+} g(x, y) \]

Often \( h(x, y) \) is a low-pass type filter. (e.g. Motion blur)

\[
\hat{f}(x, y) = g(x, y) * h_r(x, y) = (f(x, y) * h(x, y) + n(x, y)) * h_r(x, y) \\
= f(x, y) * h(x, y) * h_r(x, y) + n(x, y) * h_r(x, y)
\]

NOISE AMPLIFICATION

Image Restoration: Basics of the Frequency Domain Approaches
**Frequency Domain Formulation:**

\[
\text{FT} = H(\omega_x, \omega_y)
\]
\[
\text{FT} = F(\omega_x, \omega_y)
\]
\[
\text{FT} = N(\omega_x, \omega_y)
\]
\[
\text{FT} = G(\omega_x, \omega_y)
\]

\[G_f = H_f F_f + N_f\]

**Point-by-point multiply**

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**Basic Inversion Idea**

\[G_f = H_f F_f + N_f\]

Neglect the noise

\[H_f^{-1} G_f = F_f + H_f^{-1} N_f\]

Element-by-Element inverse

Restored

\[H\]

Inverse filter
Avoiding Singularities

Note: If $H$ has zero elements, then at those frequencies, the inverse filter does not exist. Thus use the following inverting equation instead:

$$\frac{H_f^*}{H_f^* H_f + C} \approx \begin{cases} \frac{H_f^{-1}}{H_f^2} & \text{for } |H_f^2| \gg C \\ \frac{1}{C} H_f^* & \text{for } |H_f^2| \ll C \end{cases}$$

Scales image by $1/C$ at high frequencies

A Simple Example

[Graph showing normalized Gaussian blur $H_f$ and increasing $C$]
Restoration Example

Restored \( f \):

Inverse Filter Used:

\[ \| \hat{F}_f - F_f \|^2 \rightarrow \]

Error = 1264 \hspace{1cm} Error = 247 \hspace{1cm} Error = 2046

Larger C suppresses high frequency response in the inverse filter.

WHAT IS THE BEST CHOICE FOR C?

Choice of the Parameter C

This is not a practical way of finding the best C!!
Drawbacks of this approach

• Finding the best $C$ is not simple.

• In the proposed strategy, $C$ is constant for all frequencies!! Maybe we can gain something by using a frequency dependent value.

• For high frequencies where the noise is dominant (over the signal), the image is amplified by $1/C >> 1$. This is silly, since we amplify mostly noise!

A Different Strategy

Previous Strategy - Based on $H_f$:
Where $H_f$ is high, apply exact inverse, and where low, apply $H_f/C$

New Strategy - Based on $[\text{Signal/Noise}] \times H_f$:
$|H_f| \text{SNR} >> 1$ - inverse
$|H_f| \text{SNR} << 1$ - 0

$$\text{Signal/Noise} = \text{SNR}^2 = \frac{|F(\omega_x, \omega_y)|^2}{\sigma^2}$$
The Wiener Filter

SNR is used in the following manner:

\[
\frac{H^*_f}{H^*_f H_f + C \cdot \text{SNR}^{-2}} =
\begin{cases}
H_f^{-1}, & \text{when } |H_f|^2 \cdot \text{SNR}^2 >> C \\
\frac{\text{SNR}^2}{C} H^*_f, & \text{when } |H_f|^2 \cdot \text{SNR}^2 << C
\end{cases}
\]

Simple Example
Choice of SNR

SNR is not known!

Assume white noise with unit variance & radially decaying signal:

\[ N^2(r, \theta) = 1 \]

\[ F^2(r, \theta) = r^{-\rho} \quad \rho \approx 2 \]

Polar Coordinates in Freq. Domain

Restoration Example

Error = 106 (compared to the 247 in the previous approach)
Is this the best one can do?

NO!