Chapter 7

7.1 Relative permittivity and polarizability

a. Show that the local field is given by

\[ E_{\text{loc}} = E \left( \frac{\varepsilon_r + 2}{3} \right) \]

Local field

b. Amorphous selenium (a-Se) is a high resistivity semiconductor that has a density of approximately 4.3 g cm\(^{-3}\) and an atomic number and mass of 34 and 78.96 g/mol respectively. Its relative permittivity at 1 kHz has been measured to be 6.7. Calculate the relative magnitude of the local field in a-Se. Calculate the polarizability per Se atom in the structure. What type of polarization is this? How will \( \varepsilon_r \) depend on the frequency?

c. If the electronic polarizability of an isolated atom is given by

\[ \alpha_e \approx 4\pi\varepsilon_0 r_o^3 \]

where \( r_o \) is the radius of the atom, then calculate the electronic polarizability of an isolated Se atom, which has \( r_o = 0.12 \) nm, and compare your result with that for an atom in a-Se. Why is there a difference?

Solution

a. The polarization, \( P \), is given by:

\[ P = (\varepsilon_0 [\varepsilon_r - 1])E \]

where \( E \) is the electric field.

The local field \( E_{\text{loc}} \) is given by:

\[ E_{\text{loc}} = E + \frac{P}{3\varepsilon_0} \]

Substitute for \( P \):

\[ E_{\text{loc}} = E + \frac{\varepsilon_0 [\varepsilon_r - 1]}{3\varepsilon_0}E = \left( 1 + \frac{\varepsilon_r - 1}{3} \right)E = \left( \frac{3 + \varepsilon_r - 1}{3} \right)E \]

\[ \therefore \quad E_{\text{loc}} = \left( \frac{\varepsilon_r + 2}{3} \right)E \]

b. The relative magnitude of the local field refers to the local field compared to the applied field, i.e.: \( \frac{E_{\text{loc}}}{E} \). Therefore, with \( \varepsilon_r = 6.7 \):

\[ \frac{E_{\text{loc}}}{E} = \left( \frac{\varepsilon_r + 2}{3} \right) = \left( \frac{6.7 + 2}{3} \right) = 2.9 \]

If \( D \) is the density then the concentration of Se atoms \( N \) is
\[ N = \frac{DN_i}{M_{at}} = \left( \frac{4.3 \times 10^3 \text{ kg/m}^3}{78.96 \times 10^{-3} \text{ kg/mol}} \right) \left( 6.022 \times 10^{23} \text{ mol}^{-1} \right) \]
\[ = 3.279 \times 10^{28} \text{ m}^{-3} \]

The Clausius-Mossotti equation relates the relative permittivity to the electronic polarizability, \( \alpha_e \):

\[ \frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N}{3\varepsilon_o} \alpha_e \]

\[ \therefore \]

\[ \alpha_e = \frac{3\varepsilon_o (\varepsilon_r - 1)}{N(\varepsilon_r + 2)} = \frac{3(8.854 \times 10^{-12} \text{ F/m})(6.7 - 1)}{(3.279 \times 10^{28} \text{ m}^{-3})(6.7 + 2)} \]

\[ \therefore \]

\[ \alpha_e = 5.31 \times 10^{-40} \text{ F m}^2 \]

This would be a type of electronic polarization, as Se is a covalent solid. \( \varepsilon_r \) is flat up to optical frequencies.

c The Se atom has a radius of about \( r_o = 0.12 \text{ nm} \). Substituting into the given equation:

\[ \alpha_e' \approx 4\pi\varepsilon_o r_o^3 = 4\pi(8.854 \times 10^{-9} \text{ F/m})(0.12 \times 10^{-9} \text{ m})^3 \]

\[ \therefore \]

\[ \alpha_e' \approx 1.92 \times 10^{-40} \text{ F m}^2 \]

Comparing this value and our previous value:

\[ \frac{\alpha_e}{\alpha_e'} = \frac{5.30 \times 10^{-40} \text{ F m}^2}{1.92 \times 10^{-40} \text{ F m}^2} = 2.76 \]

The observed polarizability per Se atom in the solid is 2.8 times greater than the polarizability of the isolated Se atom. In the solid, valence electrons are involved in bonding and these electrons contribute to electronic bond polarization (the field can displace these electrons).

### 7.2 Relative permittivity, bond strength, bandgap and refractive index

Diamond, silicon, and germanium are covalent solids with the same crystal structure. Their relative permittivities are shown in Table 7Q2-1.

a. Explain why \( \varepsilon_r \) increases from diamond to germanium.

b. Calculate the polarizability per atom in each crystal and then plot polarizability against the elastic modulus \( Y \) (Young’s modulus). Should there be a correlation?

c. Plot the polarizability from part b against the bandgap energy, \( E_g \). Is there a relationship?

d. Show that the refractive index \( n \) is \( \sqrt{\varepsilon_r} \). When does this relationship hold and when does it fail?

e. Would your conclusions apply to ionic crystals such as NaCl?

**Table 7Q2-1**  Properties of diamond, Si, and Ge

<table>
<thead>
<tr>
<th></th>
<th>( \varepsilon_r )</th>
<th>( M_{at} )</th>
<th>Density (g cm(^{-3}))</th>
<th>( \alpha_e )</th>
<th>( Y ) (GPa)</th>
<th>( E_g ) (eV)</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond</td>
<td>5.8</td>
<td>12</td>
<td>3.52</td>
<td>827</td>
<td>5.5</td>
<td>2.42</td>
<td></td>
</tr>
<tr>
<td>Si</td>
<td>11.9</td>
<td>28.09</td>
<td>2.33</td>
<td>190</td>
<td>1.12</td>
<td>3.45</td>
<td></td>
</tr>
<tr>
<td>Ge</td>
<td>16</td>
<td>72.61</td>
<td>5.32</td>
<td>75.8</td>
<td>0.67</td>
<td>4.09</td>
<td></td>
</tr>
</tbody>
</table>
Solution

a In diamond, Si, and Ge, the polarization mechanism is electronic (bond). There are two factors that increase the polarization. First is the number of electrons available for displacement and the ease with which the field can displace the electrons. The number of electrons in the core shells increases from diamond to Ge. Secondly, and most importantly, the bond strength per atom decreases from diamond to Ge, making it easier for valence electrons in the bonds to be displaced.

b For diamond, atomic concentration \( N \) is:

\[
N = \frac{DN_{A}}{M_{at}} = \frac{3.52 \times 10^3 \text{ kg/m}^3 \left(6.022 \times 10^{23} \text{ mol}^{-1}\right)}{12 \times 10^{-3} \text{ kg/mol}} = 1.766 \times 10^{29} \text{ m}^{-3}
\]

The polarizability can then be found from the Clausius-Mossotti equation:

\[
\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N}{3\varepsilon_o} \alpha_e
\]

\[
\therefore \quad \alpha_e = \frac{3\varepsilon_o (\varepsilon_r - 1)}{N(\varepsilon_r + 2)} = \frac{3(8.854 \times 10^{-12} \text{ F/m})(5.8 - 1)}{(1.766 \times 10^{29} \text{ m}^3)(5.8 + 2)}
\]

\[
\therefore \quad \alpha_e = 9.256 \times 10^{-41} \text{ F m}^2
\]

The polarizability for Si and Ge can be found similarly, and are summarized in Table 7Q2-2:

Table 7Q2-2  Polarizability values for diamond, Si and Ge

<table>
<thead>
<tr>
<th></th>
<th>( N ) (m(^{-3}))</th>
<th>( \alpha_e ) (F m(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond</td>
<td>1.766 \times 10^{29} m(^{-3})</td>
<td>9.256 \times 10^{-41} F m(^2)</td>
</tr>
<tr>
<td>Si</td>
<td>4.995 \times 10^{28} m(^{-3})</td>
<td>4.170 \times 10^{-40} F m(^2)</td>
</tr>
<tr>
<td>Ge</td>
<td>4.412 \times 10^{28} m(^{-3})</td>
<td>5.017 \times 10^{-40} F m(^2)</td>
</tr>
</tbody>
</table>

Figure 7Q2-1  Plot of polarizability per atom versus Young’s modulus.

\[ \alpha = 5.311 \times 10^{-40} - (5.327 \times 10^{-43})Y \]

Correlation coefficient = 0.9969
As the polarization mechanism in these crystals is due to electronic bond polarization, the displacement of electrons in the covalent bonds depends on the flexibility or elasticity of these bonds and hence also depends on the elastic modulus.

c
\[
\alpha = 5.325 \times 10^{-40} - (8.0435 \times 10^{-41})E_g \\
\text{Correlation coefficient} = 0.9873
\]

**Figure 7Q2-2** Plot of polarizability versus bandgap energy.

There indeed seems to be a linear relationship between polarizability and bandgap energy.

d
To facilitate this proof, we can plot a graph of refractive index, \( n \), versus relative permittivity, \( \varepsilon_r \).

The log-log plot exhibits a straight line through the three points. The best fit line is \( n = A\varepsilon_r^x \) (Correlation coefficient is 0.9987) where \( x = 0.513 \approx 1/2 \) and \( A = \exp(-0.02070) \approx 1 \). Thus \( n = \sqrt{\varepsilon_r} \).

The refractive index \( n \) is an optical property that represents the speed of a light wave, or an electromagnetic wave, through the material (\( v = c/n \)). The light wave is a high frequency electromagnetic wave where the frequency is of the order of \( 10^{14} \text{ to } 10^{15} \) Hz (\( f_{\text{optical}} \)). \( n \) and polarizability (or \( \varepsilon_r \)) will be related if the polarization can follow the field oscillations at this frequency (\( f_{\text{optical}} \)). This will be the case in electronic polarization because electrons are light and rapidly respond
to the fast oscillations of the field. The relationship between \( n \) and \( \varepsilon_r \) will not hold if we take \( \varepsilon_r \) at a low frequency (\( \ll f_{\text{optical}} \)) where other slow polarization contributions (such as ionic polarization, dipolar polarization, interfacial polarization) also contribute to \( \varepsilon_r \).

e \quad n = \sqrt{\varepsilon_r} \quad \text{would apply to ionic crystals if} \quad \varepsilon_r \quad \text{is taken at the corresponding optical frequency rather than at frequencies below} \quad f_{\text{optical}}. \quad \text{Tabulated data for ionic crystals typically quote} \quad \varepsilon_r \quad \text{that includes ionic polarization and hence this data does NOT conform to} \quad n = \sqrt{\varepsilon_r}.

Chapter 8

8.1 Inductance of a long solenoid

Consider the very long (ideally infinitely long) solenoid shown in Figure 8Q1-1. If \( r \) is the radius of the core and \( \ell \) is the length of the solenoid, then \( \ell >> r \). The total number of turns is \( N \) and the number of turns per unit length is \( n = \frac{N}{\ell} \). The current through the coil wires is \( I \). Apply Ampere's law around \( C \), which is the rectangular circuit \( PQRS \), and show that

\[
B \approx \mu_0 \mu_r n I
\]

Further, show that the inductance is

\[
L \approx \mu_0 \mu_r n^2 V_{\text{core}}
\]

**Inductance of long solenoid**

where \( V_{\text{core}} \) is the volume of the core. How would you increase the inductance of a long solenoid?

What is the approximate inductance of an air-cored solenoid with a diameter of 1 cm, length of 20 cm, and 500 turns? What is the magnetic field inside the solenoid and the energy stored in the whole solenoid when the current is 1 A? What happens to these values if the core medium has a relative permeability \( \mu_r \) of 600?

---

**Solution**

We use Ampere's law in Equation 8.15 (in the textbook). Consider Figure 8Q1-2. If \( H \) is the field along a small length \( d\ell \) along a closed path \( C \), then around \( C \), \( \int H d\ell = \text{total threaded current} = I_{\text{total}} = NI \).
Assume that the solenoid is infinitely long. The rectangular loop PQRS has \( n(PQ) \) number of turns where \( n \) is the number of turns per unit length or \( n = N/l \) (See Figure 8Q1-1). The field is only inside the solenoid and only along the PQ direction (long solenoid assumption) and therefore the field along QR, RS and SP is zero. Assume that the field \( H \) is uniform across the solenoid core cross section. Then the path integral of the magnetic field intensity \( H \) around PQRS is simply \( Hl = H(PQ) \).

Ampere’s law \( \int Hd\ell = I_{\text{total}} \) is then
\[
H(PQ) = I(nPQ)
\]
i.e.
\[
H = nI
\]

The dimensions of the solenoid are such that length >> diameter. We can assume that \( H \) field is relatively uniform at all points inside the solenoid. Note: The approximate equality sign in the text (equation for \( B \)) is due to the fact that we assumed \( H \) is uniform across the core and, further, along the whole length of the solenoid from one end to the other. The ends of the solenoid will have different fields (lower). Let \( A \) be the cross-sectional area of the solenoid. The magnetic field \( B \), the flux \( \Phi \) and hence the inductance \( L \) are
\[
B = \mu_0\mu_r H \approx \mu_0\mu_r nI
\]
\[
\therefore \quad \Phi = BA \approx \mu_0\mu_r nAI = \mu_0\mu_r (N/l)AI
\]
and
\[
L = (N\Phi)/I = N[\mu_0\mu_r (N/l)AI]/I = \mu_0\mu_r (N^2/l)A = \mu_0\mu_r n^2 (\ell A)
\]
\[
\therefore \quad L = \mu_0\mu_r n^2 V_{\text{core}}
\]
where \( V_{\text{core}} \) is the volume of the core. Inductance depends on \( n^2 \), where \( n \) is the number of turns per unit length, on the relative permeability \( \mu_r \) and on the volume of the core containing the magnetic flux. For a given volume inductor, \( L \) can be increased by using a higher \( \mu_r \) material or increasing \( n \), e.g. thinner wire to get more turns per unit length (not so thin that the skin effect diminishes the \( Q \)-factor, quality factor; see §2.8 in the textbook). The theoretical inductance of the coil is
\[
L = (4\pi \times 10^{-7} \text{ H/m})(1)(500)/(0.2 \text{ m})^2(0.2 \text{ m})(\pi)(0.01 \text{ m})/(2) = 3.14 \times 10^{-3} \text{ T}
\]
\[
\therefore \quad L = 1.23 \times 10^{-4} \text{ H or } 0.123 \text{ mH}
\]
and
\[
B \approx (4\pi \times 10^{-7} \text{ Wb A}^{-1} \text{ m}^{-1})(1)(500)/(0.2 \text{ m})(1 \text{ A}) = 3.14 \times 10^{-3} \text{ T}
\]
The energy per unit volume is,
\[
E_{\text{vol}} = B^2/(2\mu_0) = (3.14 \times 10^{-3} \text{ T})^2/[2(4\pi \times 10^{-7} \text{ Wb A}^{-1} \text{ m}^{-1})]
\]
\[
\therefore \quad E_{\text{vol}} = 3.92 \text{ J/m}^3
\]
The total energy stored is then,
\[ E_{\text{tot}} = E_{\text{vol}} \text{(Length} \times \text{Area)} = (3.92 \text{ J/m}^3)\pi \left( \frac{0.01 \text{ m}}{2} \right)^2 (0.2 \text{ m}) = 61.6 \mu J \]

Suppose that \( \mu_r = 600 \) and suppose that the core does not saturate (an ideal ferromagnetic material) then,
\[ L \approx (1.23 \times 10^{-4} \text{ H})(600) = 0.0738 \text{ H} \]
\[ B \approx (3.14 \times 10^{-3} \text{ T})(600) = 1.88 \text{ T} \]
and
\[ E_{\text{vol}} \approx \frac{(1.88 \text{ T})^2}{2(4\pi \times 10^{-7} \text{ Wb/A} \cdot \text{m})(600)} = 2344 \text{ J/m}^3 \]
so that
\[ E_{\text{tot}} = (E_{\text{vol}})(\text{Volume}) = 36.8 \text{ mJ} \]

This is a dramatic increase and shows the virtue of using a magnetic core material for increasing the inductance and the stored magnetic energy.

**8.2 Magnetization**

Consider a long solenoid with a core that is an iron alloy (see Problem 8.1 for the relevant formulas). Suppose that the diameter of the solenoid is 2 cm and the length of the solenoid is 20 cm. The number of turns on the solenoid is 200. The current is increased until the core is magnetized to saturation at about \( I = 2 \text{ A} \) and the saturated magnetic field is 1.5 T.

a. What is the magnetic field intensity at the center of the solenoid and the applied magnetic field, \( \mu_r H \), for saturation?

b. What is the saturation magnetization \( M_{\text{sat}} \) of this iron alloy?

c. What is the total magnetization current on the surface of the magnetized iron alloy specimen?

d. If we were to remove the iron-alloy core and attempt to obtain the same magnetic field of 1.5 T inside the solenoid, how much current would we need? Is there a practical way of doing this?

**Solution**

a. Applying Ampere’s law or \( H\ell = NI \) we have,
\[ H = \frac{NI}{\ell} = \frac{(200)(2 \text{ A})}{0.2 \text{ m}} \]
Since \( I = 2 \text{ A} \) gives saturation, corresponding magnetizing field is
\[ H_{\text{sat}} \approx 2000 \text{ A/m} \]
Suppose the applied magnetic field is the magnetic field in the toroid core in the absence of material. Then
\[ B_{\text{app}} = \mu_o H_{\text{sat}} = \left( 4\pi \times 10^{-7} \text{ Wb/} \text{A} \cdot \text{m} \right)(2000 \text{ A/m}) \]
\[ \therefore B_{\text{app}} = 2.51 \times 10^{-3} \text{ T} \]

b. Apply
\[ B_{\text{sat}} = \mu_o \left( M_{\text{sat}} + H_{\text{sat}} \right) \]
\[ M_{\text{sat}} = \frac{B_{\text{sat}}}{\mu_0} - H_{\text{sat}} = \frac{1.5 \text{ T}}{4\pi \times 10^{-7} \text{ Wb} \text{ A}^{-1} \text{ m}^{-1}} - 2000 \text{ A/m} \]

\[ M_{\text{sat}} \approx 1.19 \times 10^6 \text{ A/m} \]

c Since \( M \) is the magnetization current per unit length,

\[ I_m = M_{\text{sat}} \approx 1.19 \times 10^6 \text{ A/m} \]

Then \( I_{\text{surface}} \) = Total circulating surface current:

\[ I_{\text{surface}} = I_m l = (1.19 \times 10^6 \text{ A/m})(0.2 \text{ m}) = 2.38 \times 10^5 \text{ A} \]

Note that the actual current in the wires, 2 A is negligible compared with \( I_{\text{surface}} \).

d Apply, \( B \approx \mu_0 I \) (for air)

\[ I \approx \frac{1.5 \text{ T}}{\left(4\pi \times 10^{-7} \text{ Wb} \text{ A}^{-1} \text{ m}^{-1}\right) \left(\frac{200}{0.2 \text{ m}}\right)} = 1194 \text{ A} \]

Not very practical in every day life! Perhaps this current (thus field \( B = 1.5 \text{ T} \)) could be achieved by using a superconducting solenoid.