Chapter 6

Sections 6-1 to 6-6: Faraday’s Law and its Applications

Problem 6.1  The switch in the bottom loop of Fig. 6-17 (P6.1) is closed at $t = 0$ and then opened at a later time $t_1$. What is the direction of the current $I$ in the top loop (clockwise or counterclockwise) at each of these two times?

![Diagram of electrical circuit](image)

Figure P6.1: Loops of Problem 6.1.

Solution: The magnetic coupling will be strongest at the point where the wires of the two loops come closest. When the switch is closed the current in the bottom loop will start to flow clockwise, which is from left to right in the top portion of the bottom loop. To oppose this change, a current will momentarily flow in the bottom of the top loop from right to left. Thus the current in the top loop is momentarily clockwise when the switch is closed. Similarly, when the switch is opened, the current in the top loop is momentarily counterclockwise.

Problem 6.2  The loop in Fig. 6-18 (P6.2) is in the $x$-$y$ plane and $B = 2B_0 \sin \omega t$ with $B_0$ positive. What is the direction of $I$ ($\hat{\phi}$ or $-\hat{\phi}$) at (a) $t = 0$, (b) $\omega t = \pi/4$, and (c) $\omega t = \pi/2$?

Solution: $I = V_{\text{emf}}/R$. Since the single-turn loop is not moving or changing shape with time, $V_{\text{emf}}^m = 0$ V and $V_{\text{emf}} = V_{\text{emf}}^r$. Therefore, from Eq. (6.8),

$$I = \frac{V_{\text{emf}}^r}{R} = -\frac{1}{R} \int_S \frac{\partial B}{\partial t} \cdot ds.$$

If we take the surface normal to be $+\hat{z}$, then the right hand rule gives positive flowing current to be in the $+\hat{\phi}$ direction.

$$I = -\frac{A}{R} \frac{\partial}{\partial t} B_0 \sin \omega t = \frac{-AB_0\omega}{R} \cos \omega t \quad \text{(A)},$$
where $A$ is the area of the loop.

(a) $A$, $\omega$ and $R$ are positive quantities. At $t = 0$, $\cos \omega t = 1$ so $I < 0$ and the current is flowing in the $-\hat{\phi}$ direction (so as to produce an induced magnetic field that opposes $B$).

(b) At $\omega t = \pi/4$, $\cos \omega t = \sqrt{2}/2$ so $I < 0$ and the current is still flowing in the $-\hat{\phi}$ direction.

(c) At $\omega t = \pi/2$, $\cos \omega t = 0$ so $I = 0$. There is no current flowing in either direction.

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**Problem 6.3** A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the $x$- or $y$-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) $B = 210e^{-2t}$ (T),

(b) $B = 210\cos x \cos 10^3t$ (T),

(c) $B = 210\cos x \sin 2y \cos 10^3t$ (T).

**Solution:** Since the coil is not moving or changing shape, $V_{\text{emf}} = 0$ V and $V_{\text{emf}} = V_{\text{emf}}^m$. From Eq. (6.6),

$$V_{\text{emf}} = -N \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = -N \frac{d}{dt} \int_{-0.125}^{0.125} \int_{-0.125}^{0.125} \mathbf{B} \cdot (\hat{z} \, dx \, dy),$$

where $N = 100$ and the surface normal was chosen to be in the $+\hat{z}$ direction.

(a) For $B = 210e^{-2t}$ (T),

$$V_{\text{emf}} = -100 \frac{d}{dt} (10e^{-2t}(0.25)^2) = 125e^{-2t} \quad (V).$$
which yields \( B_0 = 1.06 \) (nA/m).

**Problem 6.6** The square loop shown in Fig. 6-19 (P6.6) is coplanar with a long, straight wire carrying a current

\[
i(t) = 2.5 \cos 2\pi \times 10^4 t \quad \text{(A)}.
\]

(a) Determine the emf induced across a small gap created in the loop.

(b) Determine the direction and magnitude of the current that would flow through a 4-\( \Omega \) resistor connected across the gap. The loop has an internal resistance of 1 \( \Omega \).

![Figure P6.6: Loop coplanar with long wire (Problem 6.6).](image)

**Solution:**

(a) The magnetic field due to the wire is

\[
B = \frac{\mu_0 I}{2\pi r} = -\hat{\phi} \frac{\mu_0 l}{2\pi y},
\]

where in the plane of the loop, \( \hat{\phi} = -\hat{r} \) and \( r = y \). The flux passing through the loop
is

\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{5 \text{ cm}}^{15 \text{ cm}} \left( -\mathbf{\hat{x}} \frac{\mu_0 I}{2\pi y} \right) \cdot [-\mathbf{\hat{x}} 10 \text{ (cm)}] \ dy
\]
\[
= \frac{\mu_0 I}{2\pi} \ln \frac{15}{5} 
\]
\[
= 4\pi \times 10^{-7} \times 2.5 \cos(2\pi \times 10^4 t) \times 10^{-1} \times \frac{1}{2\pi} 
\]
\[
= 0.55 \times 10^{-7} \cos(2\pi \times 10^4 t) \text{ (Wb)}.
\]

\[
V_{\text{emf}} = -\frac{d\Phi}{dt} = \frac{0.55 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7}}{5} 
\]
\[
= 3.45 \times 10^{-3} \sin(2\pi \times 10^4 t) \text{ (V)}.
\]

(b)

\[
I_{\text{ind}} = \frac{V_{\text{emf}}}{4+1} = \frac{3.45 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 0.69 \sin(2\pi \times 10^4 t) \text{ (mA)}.
\]

At \( t = 0 \), \( \mathbf{B} \) is a maximum, it points in \(-\mathbf{\hat{x}}\)-direction, and since it varies as \( \cos(2\pi \times 10^4 t) \), it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.

**Problem 6.7** The rectangular conducting loop shown in Fig. 6.20 (P6.7) rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

\[
\mathbf{B} = \mathbf{\hat{z}} 50 \text{ (mT)}.
\]

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

**Solution:**

\[
\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \mathbf{\hat{z}} 50 \times 10^{-3} \cdot \mathbf{\hat{z}} (2 \times 3 \times 10^{-4}) \cos \phi(t) = 3 \times 10^{-5} \cos \phi(t),
\]
\[
\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \text{ (rad/s)},
\]
\[
\Phi = 3 \times 10^{-5} \cos(200\pi t) \text{ (Wb)},
\]
\[
V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \text{ (V)},
\]
\[
I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \text{ (mA)}.
\]
From Eq. (6.24),

\[
V_{12} = V_{\text{emf}} = \int_{2}^{1} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{r=0.5}^{0} (\hat{\phi}6\pi r \times \hat{z}3 \times 10^{-4}) \cdot \hat{r} \, dr
\]

\[
= 18\pi \times 10^{-4} \int_{r=0.5}^{0} r \, dr
\]

\[
= 9\pi \times 10^{-4} r^2 \bigg|_{0.5}^{0}
\]

\[
= -9\pi \times 10^{-4} \times 0.25 = -0.707 \quad (\mu V).
\]

**Problem 6.10** The loop shown in Fig. 6-22 (P6.10) moves away from a wire carrying a current \( I_1 = 10 \) (A) at a constant velocity \( \mathbf{u} = \hat{y}5 \) (m/s). If \( R = 10 \) \( \Omega \) and the direction of \( I_2 \) is as defined in the figure, find \( I_2 \) as a function of \( y_0 \), the distance between the wire and the loop. Ignore the internal resistance of the loop.

**Solution:** Assume that the wire carrying current \( I_1 \) is in the same plane as the loop. The two identical resistors are in series, so \( I_2 = V_{\text{emf}}/2R \), where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

\[
V_{\text{emf}} = V_{\text{emf}}^m = \oint_{C} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.
\]

The magnetic field \( \mathbf{B} \) is created by the wire carrying \( I_1 \). Choosing \( \hat{z} \) to coincide with the direction of \( I_1 \), Eq. (5.30) gives the external magnetic field of a long wire to be

\[
\mathbf{B} = \frac{\mu_0 I_1}{2\pi r}.
\]
For positive values of \( y_0 \) in the \( y-z \) plane, \( \hat{y} = \hat{r} \), so

\[
\mathbf{u} \times \mathbf{B} = \hat{y}|u| \times \mathbf{B} = \hat{r}|u| \times \frac{\mu_0 I_1}{2\pi r} = \frac{\mu_0 I_1 u}{2\pi r} .
\]

Integrating around the four sides of the loop with \( dl = \hat{z} dz \) and the limits of integration chosen in accordance with the assumed direction of \( I_2 \), and recognizing that only the two sides without the resistors contribute to \( V_{\text{emf}}^m \), we have

\[
V_{\text{emf}}^m = \int_0^{0.2} \left( \frac{\mu_0 I_1 u}{2\pi r} \right) \left| \left. \cdot \hat{z} \right|_{r=y_0} \right| + \int_{0.2}^0 \left( \frac{\mu_0 I_1 u}{2\pi r} \right) \left| \left. \cdot \hat{z} \right|_{r=y_0+0.1} \right| \cdot (dz) \\
= 4\pi \times 10^{-7} \times 10 \times 5 \times 0.2 \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \\
= 2 \times 10^{-6} \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad (V),
\]

and therefore

\[
I_2 = \frac{V_{\text{emf}}^m}{2R} = 100 \left( \frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad (\text{nA}).
\]

**Problem 6.11** The conducting cylinder shown in Fig. 6-23 (P6.11) rotates about its axis at 1,200 revolutions per minute in a radial field given by

\[
\mathbf{B} = \hat{r} \mathcal{E} \quad (T).
\]
\[ D = \varepsilon_0 E \]
\[ = \varepsilon_0 \varepsilon_0 E \]
\[ = -\hat{r} 9 \times 8.85 \times 10^{-12} \times \frac{144.3}{r} \sin(120\pi r) \]
\[ = -\hat{r} \frac{1.15 \times 10^{-8}}{r} \sin(120\pi r) \text{ (C/m}^2\text{)}.
\]

The displacement current flows between the conductors through an imaginary cylindrical surface of length \( l \) and radius \( r \). The current flowing from the outer conductor to the inner conductor along \(-\hat{r}\) crosses surface \( S \) where

\[ S = -\hat{r} 2\pi rl. \]

Hence,

\[
I_d = \frac{\partial D}{\partial t} \cdot S = -\hat{r} \frac{\partial}{\partial t} \left( \frac{1.15 \times 10^{-8}}{r} \sin(120\pi r) \right) \cdot (-\hat{r} 2\pi rl)
\]
\[ = 1.15 \times 10^{-8} \times 120\pi \times 2\pi l \cos(120\pi r) \]
\[ = 1.63 \cos(120\pi r) \text{ (\muA)}.
\]

Alternatively, since the coaxial capacitor is lossless, its displacement current has to be equal to the conduction current flowing through the wires connected to the voltage sources. The capacitance of a coaxial capacitor is given by (4.116) as

\[ C = \frac{2\pi \varepsilon l}{\ln\left(\frac{b}{a}\right)}. \]

The current is

\[
I = C \frac{dV}{dt} = \frac{2\pi \varepsilon l}{\ln\left(\frac{b}{a}\right)} [120\pi \times 100\cos(120\pi r)] = 1.63 \cos(120\pi r) \text{ (\muA)},
\]

which is the same answer we obtained before.

\[ \textbf{Problem 6.16} \]: The parallel-plate capacitor shown in Fig. 6-25 (P6.16) is filled with a lossy dielectric material of relative permittivity \( \varepsilon_r \) and conductivity \( \sigma \). The separation between the plates is \( d \) and each plate is of area \( A \). The capacitor is connected to a time-varying voltage source \( V(t) \).

(a) Obtain an expression for \( I_c \), the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.
Figure P6.16: Parallel-plate capacitor containing a lossy dielectric material (Problem 6.16).

(b) Obtain an expression for $I_d$, the displacement current flowing inside the capacitor.

(c) Based on your expression for parts (a) and (b), give an equivalent-circuit representation for the capacitor.

(d) Evaluate the values of the circuit elements for $A = 2 \text{ cm}^2$, $d = 0.5 \text{ cm}$, $\varepsilon_r = 4$, $\sigma = 2.5 \text{ (S/m)}$, and $V(t) = 10\cos(3\pi \times 10^3 t) \text{ (V)}$.

Solution:

(a) 

$$R = \frac{d}{\sigma A}, \quad I_c = \frac{V}{R} = \frac{V \sigma A}{d}.$$ 

(b) 

$$E = \frac{V}{d}, \quad I_d = \frac{\partial D}{\partial t} \cdot A = \varepsilon A \frac{\partial E}{\partial t} = \frac{\varepsilon A}{d} \frac{\partial V}{\partial t}.$$ 

(c) The conduction current is directly proportional to $V$, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence,

$$R = \frac{d}{\sigma A} \quad \text{and} \quad C = \frac{\varepsilon A}{d}.$$ 

(d) 

$$R = \frac{0.5 \times 10^{-2}}{2.5 \times 2 \times 10^{-4}} = 10 \text{ \Omega}.$$
Problem 6.17 An electromagnetic wave propagating in seawater has an electric field with a time variation given by \( E = \varepsilon E_0 \cos \omega t \). If the permittivity of water is \( 81\varepsilon_0 \) and its conductivity is \( 4 \) (S/m), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies: (a) 1 kHz, (b) 1 MHz, (c) 1 GHz, (d) 100 GHz.

Solution: From Eq. (6.44), the displacement current density is given by

\[
J_d = \frac{\partial}{\partial t} D = \varepsilon \frac{\partial}{\partial t} E
\]

and, from Eq. (4.67), the conduction current is \( J = \sigma E \). Converting to phasors and taking the ratio of the magnitudes,

\[
\left| \frac{\tilde{J}}{\tilde{J}_d} \right| = \left| \frac{\sigma \tilde{E}}{j\omega \varepsilon_0 \tilde{E}} \right| = \frac{\sigma}{\omega \varepsilon_0 \varepsilon_0}.
\]

(a) At \( f = 1 \) kHz, \( \omega = 2\pi \times 10^3 \) rad/s, and

\[
\left| \frac{\tilde{J}}{\tilde{J}_d} \right| = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.
\]

The displacement current is negligible.

(b) At \( f = 1 \) MHz, \( \omega = 2\pi \times 10^6 \) rad/s, and

\[
\left| \frac{\tilde{J}}{\tilde{J}_d} \right| = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.
\]
(d)

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \approx 120\pi \sqrt{\frac{24}{\varepsilon_{\text{r}}}} = 120\pi \sqrt{\frac{2.4}{1.67}} = 451.94 \quad (\Omega), \]

\[ \vec{H} = \frac{1}{\eta} (-\hat{z}) \times \vec{E} = \frac{1}{\eta} (-\hat{z}) \times \hat{j} 10e^{j0.2t} = \hat{i} 22.13e^{j0.2t} \quad (\text{mA/m}), \]

\[ \vec{H}(z,t) = \hat{i} 22.13 \cos(\omega t + 0.2z) \quad (\text{mA/m}), \]

with \( \omega = 2\pi f = 9.54\pi \times 10^6 \text{ rad/s}. \)

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**Problem 7.4** The electric field of a plane wave propagating in a nonmagnetic material is given by

\[ \vec{E} = [\hat{j} 3 \sin(2\pi \times 10^7 t - 0.4\pi x) + \hat{k} 4 \cos(2\pi \times 10^7 t - 0.4\pi x)] \quad (\text{V/m}). \]

Determine (a) the wavelength, (b) \( \varepsilon_{\text{r}} \), and (c) \( \vec{H} \).

**Solution:**

(a) Since \( k = 0.4\pi \),

\[ \lambda = \frac{2\pi}{k} = \frac{2\pi}{0.4\pi} = 5 \text{ m}. \]

(b)

\[ u_p = \frac{\omega}{k} = \frac{2\pi \times 10^7}{0.4\pi} = 5 \times 10^7 \text{ m/s}. \]

But

\[ u_p = \frac{c}{\sqrt{\varepsilon_{\text{r}}}}. \]

Hence,

\[ \varepsilon_{\text{r}} = \left( \frac{c}{u_p} \right)^2 = \left( \frac{3 \times 10^8}{5 \times 10^7} \right)^2 = 36. \]

(c)

\[ \vec{H} = \frac{1}{\eta} \hat{k} \times \vec{E} = \frac{1}{\eta} \hat{k} \times [\hat{j} 3 \sin(2\pi \times 10^7 t - 0.4\pi x) + \hat{k} 4 \cos(2\pi \times 10^7 t - 0.4\pi x)] \]

\[ = \frac{\hat{i} 3}{\eta} \sin(2\pi \times 10^7 t - 0.4\pi x) - \frac{\hat{j} 4}{\eta} \cos(2\pi \times 10^7 t - 0.4\pi x) \quad (\text{A/m}), \]
with
\[ \eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \approx \frac{120\pi}{6} = 20\pi = 62.83 \text{ (}\Omega\text{).} \]

**Problem 7.5** A wave radiated by a source in air is incident upon a soil surface, whereupon a part of the wave is transmitted into the soil medium. If the wavelength of the wave is 30 cm in air and 15 cm in the soil medium, what is the soil's relative permittivity? Assume the soil to be a very low loss medium.

**Solution:** From \( \lambda = \lambda_0 / \sqrt{\varepsilon_r} \),
\[ \varepsilon_r = \left( \frac{\lambda_0}{\lambda} \right)^2 = \left( \frac{30}{15} \right)^2 = 4. \]

**Problem 7.6** The electric field of a plane wave propagating in a lossless, nonmagnetic, dielectric material with \( \varepsilon_r = 2.56 \) is given by
\[ E = \frac{\phi}{20} \cos(8\pi \times 10^9 t - kz) \text{ (V/m).} \]

Determine:
(a) \( f, \ u_p, \lambda, \ k, \) and \( \eta \), and
(b) the magnetic field \( \mathbf{H} \).

**Solution:**
(a)
\[ \omega = 2\pi f = 8\pi \times 10^9 \text{ rad/s,} \]
\[ f = 4 \times 10^9 \text{ Hz = 4 GHz,} \]
\[ u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.56}} = 1.875 \times 10^8 \text{ m/s,} \]
\[ \lambda = \frac{u_p}{f} = \frac{1.875 \times 10^8}{4 \times 10^9} = 4.69 \text{ cm,} \]
\[ k = \frac{2\pi}{\lambda} = \frac{2\pi}{4.69 \times 10^{-2}} = 134.04 \text{ rad/m,} \]
\[ \eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{2.56}} = \frac{377}{1.6} = 235.62 \text{ }\Omega. \]