Magnetic Circuits and Transformers (Chapter 15)

EE 70
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Agenda

• Physical model for resistors
• Physical model for capacitors
• Physical model for inductors
• Mutual inductance
• Transformers
• Diodes
Physical Model of Resistors

- $\rho$: resistivity (fundamental property of the material)
- Resistance depends on how long the object is and its cross-sectional area
- See p. 30-31

$$R = \frac{\rho L}{A}$$

<table>
<thead>
<tr>
<th>Table 1.3. Resistivity Values ($\Omega \cdot m$) for Selected Materials at 300 K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conductors</td>
</tr>
<tr>
<td>Aluminum</td>
</tr>
<tr>
<td>Carbon (amorphous)</td>
</tr>
<tr>
<td>Copper</td>
</tr>
<tr>
<td>Gold</td>
</tr>
<tr>
<td>Nichrome</td>
</tr>
<tr>
<td>Silver</td>
</tr>
<tr>
<td>Tungsten</td>
</tr>
<tr>
<td>Semiconductors</td>
</tr>
<tr>
<td>Silicon (device grade)</td>
</tr>
<tr>
<td>depends on impurity concentration</td>
</tr>
<tr>
<td>Insulators</td>
</tr>
<tr>
<td>Fused quartz</td>
</tr>
<tr>
<td>Glass (typical)</td>
</tr>
<tr>
<td>Teflon</td>
</tr>
</tbody>
</table>
Capacitance of a Parallel-Plate Capacitor

\[ C = \frac{\varepsilon A}{d} \quad A = WL \]

(derived using Gauss’ Law)

\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} \]

(vacuum permittivity)

\[ \varepsilon = \varepsilon_r \varepsilon_0 \]

Dielectric constant (value is material dependent)

<table>
<thead>
<tr>
<th>Table 3.1. Relative Dielectric Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
</tr>
<tr>
<td>Diamond</td>
</tr>
<tr>
<td>Mica</td>
</tr>
<tr>
<td>Polyester</td>
</tr>
<tr>
<td>Quartz</td>
</tr>
<tr>
<td>Silicon dioxide</td>
</tr>
<tr>
<td>Water</td>
</tr>
</tbody>
</table>
**Sample Calculation of Capacitance**

![Diagram of capacitor](image)

**Figure 3.12** Practical capacitors can be constructed by interleaving the plates with two dielectric layers and rolling them up. By staggering the plates, connection can be made to one plate at each end of the roll.

A fairly large

\[
A = WL = (10\text{cm})(20\text{cm}) = (10 \times 10^{-2} \text{m})(20 \times 10^{-2} \text{m}) = 0.02 \text{m}^2
\]

\[
d = 0.1 \text{mm} = 1 \times 10^{-4} \text{m}
\]

\[
C = \varepsilon_0 \frac{A}{d} = \left(8.854 \times 10^{-12} \frac{\text{F}}{\text{m}}\right) \left[\frac{0.02 \text{m}^2}{1 \times 10^{-4} \text{m}}\right] = 1770 \times 10^{-12} \text{F} = 1.77 \text{nF}
\]
Inductance

- Wire coiled around a magnetic core
  - Current flow $\rightarrow$ Magnetic field / flux
  - Core improves magnetic flux and can re-direct it
Flux Linkages

Magnetic flux passing through a surface area $A$:

$$\phi = \int_{A} B \cdot dA$$

For a constant magnetic flux density perpendicular to the surface:

$$= BA$$

The flux linking a coil with $N$ turns:

$$\lambda = N\phi$$
Definition of Inductance

\[ L = \frac{\text{Flux linkages}}{\text{current}} = \frac{\lambda}{i} \]

Substitute for the flux linkages using \( \lambda = N\phi \)

\[ L = \frac{N\phi}{i} \]
Reluctance

- The magnetic flux can also be defined in terms of a quantity called reluctance
  - Magnetic equivalent of resistance

\[ R_N = \frac{f}{\mu A} \]

Permeability (material dependent)

\[ \phi = \frac{Ni}{R} \]

\[ L = \frac{N\phi}{i} = \frac{N^2}{R} \]

(constant, time independent)
Faraday’s Law

Faraday’s law of magnetic induction:

\[ V = \frac{d\lambda}{dt} = \frac{d(Li)}{dt} = L \frac{di}{dt} \]

The voltage induced in a coil whenever its flux linkages are changing. Changes occur from:

- Magnetic field changing in time
- Coil moving relative to magnetic field
Mutual Inductance

- Current flowing through one coil can produce a magnetic field that can induce a current in a different coil

Self inductance for coil 1

\[ L_1 = \frac{\lambda_{1\rightarrow 1}}{i_1} = \frac{\lambda_{11}}{i_1} \]

Self inductance for coil 2

\[ L_2 = \frac{\lambda_{2\rightarrow 2}}{i_2} = \frac{\lambda_{22}}{i_2} \]

Mutual inductance between coils 1 and 2:

\[ M = \frac{\lambda_{2\rightarrow 1}}{i_1} = \frac{\lambda_{21}}{i_1} = \frac{\lambda_{1\rightarrow 2}}{i_2} = \frac{\lambda_{12}}{i_2} \]
Circuit Equations for Mutual Inductance

- The mutual inductance can increase or decrease the voltage
  - Depends on the direction of the field produced

\[
\lambda_1 = L_1 i_1 \pm M i_2
\]
\[
\lambda_2 = \pm M i_1 + L_2 i_2
\]
\[
v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}
\]
\[
v_2 = \frac{d\lambda_2}{dt} = \pm M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}
\]
Mutual Inductance

- To determine whether or not the mutual inductance increases or decreases the voltage, one of the terminals is “dotted”
  - Currents entering the dotted terminals produce aiding fluxes (based on right hand rule)

\[
\begin{align*}
  v_1 & = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\
  v_2 & = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}
\end{align*}
\]

Fields are aiding

\[
\begin{align*}
  v_1 & = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \\
  v_2 & = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}
\end{align*}
\]

Fields are opposing
Example 15.8

\[ R = 10^7 \text{ (ampere-turns)/Weber} \]

Self inductance:

\[ L_1 = \frac{\frac{N_1^2}{R}}{10^7} = \frac{100^2}{10^7} = 1mH \]

\[ L_2 = \frac{\frac{N_2^2}{R}}{10^7} = \frac{200^2}{10^7} = 4mH \]
Example 15.8

\[ \phi_1 = \frac{N_1 i_1}{R} = \frac{100i_1}{10^7} = 10^{-5} i_1 \]

(Magnetic flux from left coil)

\[ \lambda_{21} = N_2 \phi_1 = 200 \times 10^{-5} i_1 \]

(Flux linkage on right coil, due to flux from left coil)

Mutual inductance:

\[ M = \frac{\lambda_{21}}{i_1} = 2 \text{mH} \]
Example 15.8

• Flux produced by $i_2$ opposes the flux produced by $i_1$
  • Proof: use right hand rule
  • If there’s a dot on the lower terminal of the left coil, then add a dot to the upper terminal of the right coil

$$e_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$
$$e_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$
Transformers

- An application of mutual inductance
- Used to step up or step down AC voltages
Ideal Transformers

\[ v_1(t) = V_{1m} \cos(\omega t) = N_1 \frac{d\phi}{dt} \quad \text{by Faraday's Law} \]

\[ \phi(t) = \frac{1}{N_1} \int v_1(t) dt + \phi_0 = \frac{V_{1m}}{N_1} \int \cos(\omega t) dt + \phi_0 = \frac{V_{1m}}{N_1 \omega} \sin(\omega t) + \phi_0 \]
Ideal Transformers

\[ v_2(t) = N_2 \frac{d\phi}{dt} = N_2 \frac{V_{1m}}{N_1 \omega} \frac{d}{dt} \left[ \sin(\omega t) + \phi_0 \right] \]

\[ = \frac{N_2}{N_1} V_{1m} \cos(\omega t) = \frac{N_2}{N_1} v_1(t) \]

Step up voltage: \( N_2 > N_1 \)
Step down: \( N_2 < N_1 \)
Lenz’s Law

• Why is $i_2$ going from left to right?
  • Unlike Example 15.8, there is no applied voltage / current on the right side

• Polarity of the induced voltage is such that the voltage would produce a current (through an external resistance) that opposes the original change in flux linkages

Figure 15.4 When the flux linking a coil changes, a voltage is induced in the coil. The polarity of the voltage is such that if a circuit is formed by placing a resistance across the coil terminals, the resulting current produces a field that tends to oppose the original change in the field.
The magneto-motive force (mmf) applied to the core:

\[ F = N_1 i_1 - N_2 i_2 = R \phi \approx 0 \text{ since } R \approx 0 \]

\[ N_1 i_1 = N_2 i_2 \]

\[ i_2(t) = \frac{N_1}{N_2} i_1(t) \]

If the voltage is stepped up
the current is stepped down
Ideal Transformers

• Since an ideal transformer is a purely inductive circuit (no resistance), there should be no power lost
  • This means that the AC current will be stepped down if the AC voltage is stepped up
  • Net power is neither generated nor consumed by an ideal transformer

\[ p_2(t) = v_2(t)i_2(t) = \frac{N_2}{N_1} v_1(t)i_2(t) \]

\[ p_1(t) = v_1(t)i_1(t) \]

\[ p_2(t) = p_1(t) \quad \rightarrow \quad i_2(t) = \frac{N_1}{N_2} i_1(t) \]
Impedance Transformations

• Goal: transform the circuit into one where the load impedance \((Z_L)\) and transformer are merged into a single impedance \((Z_L')\)
  • Circuit elements on the secondary side are reflected to the primary side

\[
Z_L = \frac{V_2}{I_2} = \frac{(N_2 / N_1)V_1}{(N_1 / N_2)I_1}
\]

\[
= \frac{(N_2 / N_1)}{(N_1 / N_2)} Z_L' = \left(\frac{N_2}{N_1}\right)^2 Z_L'
\]

\[
Z_L' = \frac{V_1}{I_1}
\]

\[
Z_L' = \left(\frac{N_1}{N_2}\right)^2 Z_L
\]
Example 15.11

- Impedance transformations are used to find the voltages across the transformer

\[ Z_L = 10 + j20 \]

\[ Z_L' = \left( \frac{N_1}{N_2} \right)^2 Z_L = (10)^2 (10 + j20) = 1000 + j2000 \]

\[ Z_s = R_1 + Z_L' = 1000 + 1000 + j2000 \]

\[ = 2000 + j2000 = 2000\sqrt{2}\angle45 = 2828\angle45 \]
Example 15.11

\[
I_1 = \frac{V_s}{Z_s} = \frac{1000 \angle 0^\circ}{2828 \angle 45^\circ} = 0.3536 \angle -45^\circ \text{ A}
\]

\[
V_1 = I_1 Z_L = (0.3536 \angle -45^\circ)(1000 + j2000)
\]

\[
= (0.3536 \angle -45^\circ)(2236 \angle 63.43^\circ)
\]

\[
= 790.6 \angle 18.43^\circ
\]
Example 15.11

We can now calculate the current and voltage phasors on the secondary side using the turns ratio:

$$I_2 = \frac{N_1}{N_2} I_1 = 10(0.3536 \angle -45^\circ) = 3.536 \angle -45^\circ$$

$$V_2 = \frac{N_2}{N_1} V_1 = \frac{1}{10} (790.6 \angle 18.43^\circ) = 79.60 \angle 18.43^\circ$$
Example 15.12

- Circuit elements and sources can also be reflected from the primary side to the secondary side of the transformer.

\[ V'_s = \frac{N_2}{N_1} V_s = \frac{1}{10} \times 1000 \angle 0^\circ \]

Reciprocal of \( Z_L \) equation:

\[ R'_1 = \left( \frac{N_2}{N_1} \right)^2 R_1 = \left( \frac{1}{10} \right)^2 (1000) = 10 \Omega \]
Example 15.12

\[ V_2 = \frac{10 + j20}{10 + 10 + j20} \quad V_s' = \frac{10 + j20}{20 + j20} \quad (100 \angle 0^\circ) \]

\[ = \frac{\sqrt{100 + 400 \angle Tan^{-1}(20/10)}}{20\sqrt{2} \angle 45^\circ} \quad (100 \angle 0^\circ) \]

\[ = \frac{10\sqrt{5} \angle 63.43^\circ}{20\sqrt{2} \angle 45^\circ} \quad (100 \angle 0^\circ) = 79.06 \angle 18.43^\circ \]

Same result as Example 15.11
What is a Semiconductor?

• Resistivity is in between a conductor and an insulator
  – Conductor: $\rho < 10^{-2} \ \Omega\text{-cm}$ (ex. aluminum, copper)
  – Insulator: $\rho > 10^5 \ \Omega\text{-cm}$ (ex. silicon dioxide)

• Resistivity can be controlled precisely and even electrically modified

• Resistivity, doping concentration and carrier mobility can be obtained from graphs

\[
\rho = \frac{1}{\sigma} = \frac{1}{q\mu_n n + q\mu_p p}
\]
Why do different materials have different $\rho$?

• Different materials have different energy band diagrams

An electron in a single atom can only exist at certain energy levels

Energy levels split as atoms are brought together, which may result in gaps
Band Diagrams

- The allowed energy levels are so close to one another that they can be modeled as a continuous band.
- Gaps can occur between continuous bands
  - The size of the energy gap between the filled and unfilled levels determines whether you have a conductor, insulator, or semiconductor.
Fig. 5.2: A two dimensional pictorial view of the Si crystal showing covalent bonds as two lines where each line is a valence electron.

Doping

- To adjust the number of carriers in a semiconductor, impurities can be added
  - Use a nearby element in the periodic table
- Modeled in the band diagram as an Ed (donor/electron) or Ea (acceptor/hole) level
Fig. 5.9: Arsenic doped Si crystal. The four valence electrons of As allow it to bond just like Si but the fifth electron is left orbiting the As site. The energy required to release to free fifth-electron into the CB is very small.

Fig. 5.11: Boron doped Si crystal. B has only three valence electrons. When it substitutes for a Si atom one of its bonds has an electron missing and therefore a hole as shown in (a). The hole orbits around the B—site by the tunneling of electrons from neighboring bonds as shown in (b). Eventually, thermally vibrating Si atoms provides enough energy to free the hole from the B—site into the VB as shown.

Fig. 5.1: (a) A simplified two dimensional illustration of a Si atom with four hybrid orbitals, $\psi_{hyb}$. Each orbital has one electron. (b) A simplified two dimensional view of a region of the Si crystal showing covalent bonds. (c) The energy band diagram at absolute zero of temperature.
Fig. 5.8: Energy band diagrams for (a) intrinsic (b) $n$-type and (c) $p$-type semiconductors. In all cases, $np = n_i^2$
Properties of the $pn$ junction.

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Energy band diagrams for a $pn$ junction under (a) open circuit, (b) forward bias and (c) reverse bias conditions. (d) Thermal generation of electron hole pairs in the depletion region results in a small reverse current.

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