Homework 9

Reading: *Book of Proof* Chapter 11, Sections 12.1, 12.2, 12.3

Problems: Due Friday March 6, 2020 10pm PST in Canvas.

1. *(4 points)* A sequence of numbers $A_0, A_1, A_2, A_3, \ldots$ is defined by

\[
A_0 = 0 \\
A_1 = 1 \\
A_n = 8A_{n-1} - 12A_{n-2} + 3^n \text{ for } n \geq 2
\]

Prove by induction on $n$ that

\[
A_n = 6^n + 2^{n+1} - 3^{n+1}.
\]

2. *(6 points)* For each of the following relations,

- give 3 pairs of elements that are related,
- determine whether the relation is reflexive (justify your answer),
- determine whether the relation is symmetric (justify your answer) and
- determine whether the relation is transitive (justify your answer).

You must prove your answer for each of the three properties (reflexive, symmetric and transitive). Note that the first relation is on the set of integers, the second on the real numbers, and the third is on the set of non-zero integers.

(a) $R_1 = \{(a, b) : a, b \in \mathbb{Z} \text{ and } a \cdot b \geq 0\}$

(b) $R_2 = \{(x, y) : x, y \in \mathbb{R} \text{ and } \lceil x \rceil = \lfloor y \rfloor\}$

(c) $R_3 = \{(i, j) : i, j \in \mathbb{Z} - \{0\} \text{ and } i^2 = j^2\}$

3. *(6 points)* In each case below a relation between set $S$ and set $T$ is described. In each case explain why the relation is not a function from the set $S$ to the set $T$.

(a) $S$ is the set of ordered pairs of integers $(\mathbb{Z} \times \mathbb{Z})$ and $T = \mathbb{Q}$ is the set of rational numbers. An ordered pair of integers $s = (n, m)$ and a rational number $t$ are related if $t = n/m$.

(b) $S$ is the set of all California license plates in California and $T = \mathbb{N}$. A license plate $s \in S$ is related to $t \in \mathbb{N}$ when $t$ is the smallest number that appears in the license plate number on $s$.

(c) $S = \mathbb{R}$ is the set of real numbers and $T$ is the set of ordered pairs of real numbers $(\mathbb{R} \times \mathbb{R})$. A real number $s$ is related to the ordered pair of real numbers $t = (x, y)$ if $s = x + y$.

4. *(4 points)* In each case below, determine whether the function given is injective (one-to-one) and prove your answer.

Pay close attention to the domain and co-domain!

(a) $f : \mathbb{Q} \to \mathbb{Q}$ where $f(x) = (3 - 2x)/5$.

(b) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+ \text{ where } f(n, m) = n/m$. ($\mathbb{Q}^+$ is the set of positive rational numbers.)

(c) $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = \frac{1}{n}$

(d) $f : \mathbb{R} \to \mathbb{N}$ where $f(x) = \lfloor x^2 + 0.3 \rfloor$
5. (4 points) In each case determine whether the function is onto (surjective) and prove your answer.

Pay close attention to the domain and co-domain!

(a) $f : \mathbb{Z} \to \mathbb{Q}$ where $f(x) = (3 - 2x)/5$.
(b) $f : \mathbb{R} \to \mathbb{N}$ where $f(x) = \lceil x^2 + 0.3 \rceil$
(c) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ where $f(a, b) = a/b$. ($\mathbb{R}^+$ is the set of positive real numbers.)
(d) $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ where $f(a, b) = a/b$. 