Homework 7

Reading: *Book of Proof* Chapter 10

Problems: Due Friday February 21 PST in Canvas.
Use other sheets. There is not enough room for correct answers here. Show work.

1. (7 points) In each of the following give a value for $x$ greater than -1 and less than the modulus (the number after \(\text{mod}\) inside the parentheses) that satisfies the equation. You must show work to get credit for your answer.

   (a) $x \equiv -75 \pmod{11}$
   (b) $x \equiv 895 \pmod{13}$
   (c) $x \equiv 2^{88} \pmod{23}$
   (d) $x^2 \equiv 4 \pmod{13}$
   (e) $8x \equiv 11 \pmod{27}$

2. (6 points) Prove the following theorem:

   **Theorem** For an integer $n$,
   
   $n$ is an odd number if and only if $n^2 - 1$ is a multiple of 4.

3. (6 points) Prove that the two sets $A$ and $B$ below are equal.

   $A = \{ 9m - 7 : m \text{ is an integer} \}$ and $B = \{ 18k + b : k \text{ is an integer, and } b = 2 \text{ or } 11 \}$

4. Consider the two sets $S_1$ and $S_2$

   $S_1 = \{ k^4 : k \text{ is an odd integer} \}$ and $S_2 = \{ 8m + 1 : m \text{ is an integer} \}$

   (a) (3 points) Prove that $S_1$ is a subset of $S_2$.
   (b) (2 points) Prove that $S_2$ is not a subset of $S_1$.

5. (5 points) Prove the following theorem:

   **Theorem** For a natural number $n$ and a prime number $p$,
   
   $p \mid n!$ if and only if $p \leq n$.

For the induction proofs (the next 3 problems), be sure to

- Give the statement you are proving in terms of the induction variable,
- identify and prove the base case(s),
- give the inductive hypotheses,
- give the inductive conclusion (the statement you must prove in the inductive step),
- and then prove the inductive conclusion.
6. (5 points) Prove by induction that
\[ \forall n \in \mathbb{N}, \sum_{i=1}^{n} 3i^2 - i = n^2(n + 1) \]

7. (5 points) Prove by induction that
\[ \forall n \in \mathbb{N}, \sum_{m=1}^{n} \frac{m}{(m+1)!} = 1 - \frac{1}{(n+1)!} \]

8. (5 points) Prove by induction that
\[ \forall n \in \mathbb{N}, \ 5n^3 - 2n \text{ is a multiple of 3.} \]

Extra Credit Problem 1 In class Friday Feb.14 we talked about RSA encryption. For the parts a. and b. below, assume that your bank’s public key is (9, 15481). You may use a calculator for these problems. In fact, you may want to write a small program.

a. (2 points) Suppose you want to send the message consisting of the integer 11 to your bank. How would the integer 11 be encoded using your bank’s public key?

b. (4 points) In doing your discrete math homework you happened to notice that 15481 = 113 \cdot 137. With this knowledge decode the message consisting of the integer 1107 which was encoded using your bank’s public key.

Extra Credit Problem 2 In class Wednesday Feb.12 we proved that the number of primes is infinite by showing that adding 1 to the product of the first \(n\) primes results in a number that has a prime divisor not among the first \(n\) primes. We tried this for the first few values of \(n\), and the number we produced was a prime.

(2 points) Prove or disprove the following conjecture:

If \(p_1, p_2, \ldots, p_n\) are the first \(n\) primes then \(1 + p_1p_2 \cdots p_n\) is a prime.