Follow the leader if you can,
Hedge if you must

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Outline

- **Follow-the-Leader:**
  - works well for *easy* data: few leader changes, i.i.d.
  - but not robust to worst-case data
- **Exponential weights with simple tuning:**
  - robust, but does not exploit easy data
- **Second-order bounds:**
  - robust against worst case + can exploit i.i.d. data
  - but do not exploit few leader changes in general
- **FlipFlop:** robust + as good as FTL
Sequential Prediction with Expert Advice

- $K$ experts sequentially predict data $x_1, x_2, \ldots$
- Goal: predict (almost) as well as the best expert on average
- Applications:
  - online convex optimization
  - predicting electricity consumption
  - predicting air pollution levels
  - spam detection
  - ...

Set-up: Repeated Game

- Every round $t = 1, \ldots, T$
  1. Predict probability distribution $w_t = (w_{t,1}, \ldots, w_{t,K})$ on experts
  2. Observe expert losses $\ell_t = (\ell_{t,1}, \ldots, \ell_{t,K}) \in [0, 1]^K$
  3. Our loss is $w_t \cdot \ell_t = \sum_k w_{t,k} \ell_{t,k}$

Goal: minimize **regret**

$$\sum_{t=1}^{T} w_t \cdot \ell_t - L^*$$

where

$$L^* = \min_k \sum_{t=1}^{T} \ell_{t,k}$$
Follow-the-Leader

- Deterministically choose the expert that has predicted best in the past:

\[ w_{t,k^*} = 1 \text{ where } k^* = \arg \min_k \sum_{s=1}^{t-1} \ell_{t,k} \]

- Equivalently:

\[ w_t = \arg \min_w \mathbb{E}_{k \sim w} \left[ \sum_{s=1}^{t-1} \ell_{t,k} \right] \]
FTL: the Good News

- Regret bounded by nr of leader changes
- Proof sketch:
  - If the leader does not change, our loss is the same as the loss of the leader, so the regret stays the same
  - If the leader does change, our regret increases at most by 1 (range of losses)

- Works well for i.i.d. losses, because the leader changes only finitely many times w.h.p.
FTL on IID Losses

- 4 experts with Bernoulli 0.1, 0.2, 0.3, 0.4 losses
FTL Worst-case Losses

\[ \text{regret} = k \times \text{number of trials } T \]

\[ FTL \]
Exponential Weights

- **Follow-the-Leader:**
  \[ w_t = \arg \min_w E_{k \sim w} \left[ \sum_{s=1}^{t-1} \ell_{t,k} \right] \]

- **Exponential weights:** add KL divergence from uniform distribution as a regularizer
  \[ w_t = \arg \min_w E_{k \sim w} \left[ \sum_{s=1}^{t-1} \ell_{t,k} \right] + \frac{1}{\eta} D(w || u) \]

- \( \eta \to \infty \): recover FTL (aggressive learning)

- As \( \eta \) closer to 0: closer to uniform distribution (more conservative learning)
Simple Tuning: the Good News

- Worst-case optimal for $\eta = \sqrt{8 \ln(K)/T}$:
  \[
  \text{Regret} \leq \sqrt{T \ln(K)/2}
  \]

- Proof idea:
  - approximate our loss: $w_t \cdot l_t = \sum_k w_{t,k} l_{t,k}$
  - by the **mix loss**:
    \[
    m_t = \frac{-1}{\eta} \ln \sum_k w_{t,k} e^{-\eta l_{t,k}}
    \]
  - and bound the **approximation error**:
    \[
    \delta_t = w_t \cdot l_t - m_t
    \]
Simple Tuning: the Good News

our loss = mix loss + approx. error

\[ w_t \cdot \ell_t = m_t + \delta_t \]

- Cumulative mix loss is close to \( L^* \):
  \[
  L^* \leq \sum_{t=1}^{T} m_t \leq L^* + \frac{\ln K}{\eta}
  \]

- Hoeffding's bound:
  \[
  \delta_t \leq \frac{\eta}{8} \quad \text{Balances the two terms}
  \]

- Together:
  \[
  \sum_{t=1}^{T} w_t \cdot \ell_t - L^* \leq \frac{\ln K}{\eta} + \frac{\eta T}{8}
  \]

  \[
  \eta = \sqrt{8 \ln K/T} \quad \Rightarrow \quad \sqrt{T \ln(K)/2}
  \]
Lost Advantages of FTL

- Simple tuning does much worse than FTL on i.i.d. losses
Simple Tuning: the Bad News

• The bad news:
  - $\eta = \sqrt{\frac{8 \ln(K)}{T}}$ = conservative learning
  - In practice, better when learning rate does not go to 0 with $T$! [DGGS, 2013]
  - Lost advantages of FTL!

• We want to exploit **luckiness**:
  - robust against worst-case losses; but
  - if the data are `easy', we should learn faster!
Luckiness: Exploiting Easy Data

- Improvement for small losses:
  \[
  \text{Regret} = O\left(\sqrt{L^* \ln(K)}\right)
  \]

- Second-order Bounds:
  - [CBMS, 2007] and AdaHedge:
    \[
    O\left(\sqrt{\sum_t v_t \ln(K)}\right)
    \]
  - Related bound by [HK, 2008]
Luckiness: Exploiting Easy Data

- Improvement for small losses:
  \[ \text{Regret} = O\left(\sqrt{L^* \ln(K)}\right) \]

- Second-order Bounds:
  - [CBMS, 2007] and AdaHedge:
    \[ O\left(\sqrt{\sum_t v_t \ln(K)}\right) \]
  - Related bound by [HK, 2008]
    \[ O\left(\sqrt{\frac{L^*(T - L^*)}{T} \ln(K)}\right) \]
2^{nd}-order Bounds: I.I.D. Data

- Regret bound: \( O\left(\sqrt{\sum_t v_t \ln(K)}\right) \)

- For IID data, \( w_t \) concentrates fast on best expert:
  \[ \sum_t v_t \leq C \]
  Regret \( \leq C' \)
Recover FTL benefits for i.i.d. data
CBMS: Proof Idea

our loss = mix loss + approx. error
\[ w_t \cdot \ell_t = m_t + \delta_t \]

- **Cumulative mix loss is close to** \( L^* \):
  \[ L^* \leq \sum_{t=1}^{T} m_t \leq L^* + \frac{\ln K}{\eta} \]

- **Bernstein's bound**:
  \[ \delta_t \leq \frac{1}{2} \eta v_t + \text{lower order terms} \]

- **Together**: balancing
  \[ \text{Regret} \leq \frac{\ln K}{\eta} + \frac{1}{2} \eta \sum_{t=1}^{T} v_t \rightarrow O\left(\sqrt{\sum_t v_t \ln(K)}\right) \]
AdaHedge: Proof Idea

- Cumulative mix loss is close to $L^*$:
  \[ L^* \leq \sum_{t=1}^{T} m_t \leq L^* + \frac{\ln K}{\eta} \]
- No bound:
  \[ \delta_t = \delta_t \]
- Together: balancing
  \[ \eta = \frac{\ln(K)}{\sum_t \delta_t} \]

Regret \[ \leq \frac{\ln K}{\eta} + \sum_t \delta_t \rightarrow O\left(\sum_t \delta_t\right) = O\left(\sqrt{\sum_t v_t \ln K}\right) \]
AdaHedge: Proof Idea

**our loss = mix loss + approx. error**

\[ w_t \cdot \ell_t = m_t + \delta_t \]

- Cumulative mix loss is close to \( L^* \):

\[
L^* \leq \sum_{t=1}^{T} m_t \leq L^* + \frac{\ln K}{\eta}
\]

- No bound:

\[ \delta_t = \delta_t \]

- Together:

\[ \eta = \frac{\ln(K)}{\sum_t \delta_t} \]

Regret \( \leq \frac{\ln K}{\eta} + \sum_t \delta_t \overset{\text{balancing}}{\longrightarrow} O\left( \sum_t \delta_t \right) = O\left( \sqrt{\sum_t v_t \ln K} \right) \)

NB Bernstein's bound is pretty sharp, so in practice CBMS \( \approx \) AdaHedge up to constants.
Tuning $\eta$ Online

- Balancing $\eta$ in CBMS and AdaHedge depends on unknown quantities
- Solve this by changing $\eta = \eta_t$ with $t$
- Problem: $\sum_t m_t \leq L^* + \ln K / \eta$ breaks

**Lemma [KV, 2005]:** If $\eta_1 \geq \eta_2 \geq \eta_3 \geq \ldots$, then

$$\sum_{t=1}^{T} m_t \leq L^* + \ln(K) / \eta_T$$
2nd-order Bounds: the Bad News

- Do not recover FTL benefits for other `easy' data with a small number of leader changes
Luckiness: Exploiting Easy Data

• Improvement for small losses:

\[
\text{Regret} = O\left(\sqrt{L^* \ln(K)}\right)
\]

• Second-order Bounds:
  - [CBMS, 2007] and AdaHedge: \(O\left(\sqrt{\sum_t v_t \ln(K)}\right)\)
  - Related bound by [HK, 2008]

• FlipFlop:
  - “Follow the leader if you can, Hedge if you must”
  - Regret \(\leq\) best of AdaHedge and FTL
FlipFlop

- FlipFlop bound:

\[
\text{Regret} \leq \begin{cases} 
6 \cdot \text{FTL Regret} \\
3 \cdot \text{AdaHedge Regret Bound}
\end{cases}
\]

- Alternate Flip and Flop regimes
  - Flip: Tune \( \eta_t = \infty \) like FTL
  - Flop: Tune \( \eta_t \) like AdaHedge

- (No restarts of the algorithm, like in `doubling trick'!)
FlipFlop: Proof Ideas

- Alternate Flip and Flop regimes
  - Flip: Tune $\eta_t = \infty$ like FTL
  - Flop: Tune $\eta_t$ like AdaHedge

- Analysing two regimes:
  1. Relate mix loss for Flip to mix loss for Flop
  2. Keep approximation errors balanced between regimes
1. Relating Mix Losses

- We violate condition of KV-lemma:

\[ \eta_1 \geq \eta_2 \geq \eta_3 \geq \ldots \]

- But:

\[
\sum_t m_t \leq \sum_t m_t^{\text{flop}} + C \sum_{t \in \text{flop}} \delta_t \\
\leq L^* + \frac{\ln K}{\eta_T} + C \sum_{t \in \text{flop}} \delta_t \\
= L^* + (C + 1) \sum_{t \in \text{flop}} \delta_t
\]
2. Balance Approximation Errors

- Alternate regimes to keep approximation errors balanced:

\[
\sum_{t \in \text{flip}} \delta_t \propto \sum_{t \in \text{flop}} \delta_t
\]

\[
\text{Regret} = \sum_{t} m_t - L^* + \sum_{t} \delta_t \leq (C + 2) \sum_{t \in \text{flop}} \delta_t + \sum_{t \in \text{flip}} \delta_t
\]

\[
\propto \begin{cases} 
    \sum_{t \in \text{flip}} \sum_{t} \delta_t & \text{FTL Bound} \\
    \sum_{t \in \text{flop}} \sum_{t} \delta_t & \text{AdaHedge Bound}
\end{cases}
\]
Small Nr Leader Changes Again

- FlipFlop exploits easy data, AdaHedge does not
FTL Worst-case Again

![Graph showing the regret over the number of trials T for FTL, AdaHedge, and FlipFlop. The regret for FTL increases linearly with the number of trials, while the regrets for AdaHedge and FlipFlop remain relatively flat.]
Summary

- **Follow-the-Leader:**
  - works well for `easy' data: i.i.d., few leader changes
  - but not robust to worst-case data

- **Second-order bounds** (e.g. CBMS, AdaHedge):
  - robust against worst case + can exploit i.i.d. data
  - but do not exploit few leader changes in general

- **FlipFlop:** best of both worlds
Luckiness: What's Missing?

- **FlipFlop:**
  - “Follow the leader if you can, Hedge if you must”
  - Regret \( \leq \) best of AdaHedge and FTL

- But what if optimal \( \eta \) is in between AdaHedge and FTL?
- Can we compete with the best possible \( \eta \) chosen in hindsight?
• De Rooij, Van Erven, Grünwald, Kolen. *Follow the Leader If You Can, Hedge If You Must*. Accepted by the Journal of Machine Learning Research, 2013.
EXTRA SLIDES
No Need to Pre-process Losses

- Common assumption $\ell_{t,k} \in [0, 1]$ requires **translating** and **rescaling** the losses.

- CBMS:
  - Extension so this is **not necessary**. Important when range of losses is unknown!

- AdaHedge and FlipFlop:
  - Invariant under rescaling and translation of losses, so get this **for free**.
2^{nd}-order Bounds: I.I.D. Data

- Regret bound: $O\left(\sqrt{\sum_{t} v_t \ln(K)}\right)$
- If $w_t$ concentrates fast on best expert, then

$$\sum_{t} v_t \leq C \quad \Rightarrow \quad \text{Regret} \leq C'$$

- IID data:
  1. Balancing $\eta_t = \sqrt{\frac{2 \ln(K)}{\sum_{s}^{t-1} v_s}}$ is large for all $t \leq T$
  2. $w_t$ concentrates fast
  3. Then 1. also holds for $T + 1$
FlipFlop on I.I.D. Data

![Graph showing regret vs. number of trials T for different algorithms: FTL, Simple tuning, AdaHedge, FlipFlop. The graph illustrates the performance of these algorithms over varying numbers of trials.]
## Example: Spam Detection

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</tr>
</tbody>
</table>
Example: Spam Detection

- **Data:** \((x_t, y_t)\) with \(y_t \in \{0, 1\}\)
- **Predictions:** probability \(p_t \in [0, 1]\) that \(y_t = 1\)
- **Loss (probability of wrong label):**
  \[
  \ell(y_t, p_t) = \begin{cases} 
  p_t & \text{if } y_t = 0 \\
  1 - p_t & \text{if } y_t = 1 
  \end{cases}
  \]
- **Experts:** \(K\) spam detection algorithms
- **If expert \(k\) predicts \(p_{t,k}\), then \(\ell_{t,k} = \ell(y_t, p_{t,k})\)**
- **Regret:** expected nr. mistakes over expected nr. of mistakes of best algorithm
FTL: the Bad News

- Consider two trivial spam detectors (experts):
  \[ p_{t,1} = 0 \quad p_{t,2} = 1 \]

- If we deterministically choose an expert \( k^* \) (like FTL) then we could be wrong all the time:
  \[ \ell_{t,k^*} = 1 \quad \ell_{t,-k^*} = 0 \]

Regret:

- Let \( n \) denote the number of times expert 1 has loss 1. Then \( L^* = \min\{n, T - n\} \leq T/2 \)
- **Linear regret** = \( T - L^* \geq T/2 \)