Fixpoint Semantics

- An operational semantics for datalog program from fixpoint theory
- A fixpoint operator or immediate consequence operator, is used to generate new facts from existing facts.
- Fix point semantics (i.e. “the smallest solutions of a fixpoint equation involving fixpoint operator”) concludes with the semantics of a datalog program, i.e. \( P(I) \)

The immediate consequence operator

- Let \( P \) be a datalog program and \( K \) be an instance over the edbs and idbs of \( P \).
- An fact \( A \) is an immediate consequence for \( K \) and \( P \) if either \( A \in K \) for some edb \( R \) or \( A \leftarrow A_1 \ldots A_n \) is an instance (under some valuation) of some rules in \( P \) and each \( A_i \), \( i \in [1,n] \), is in \( K \).

Immediate consequence operator of \( P \) is denoted as \( T_p \)

\[
T_p : \text{inst}(\text{sch}(P)) \rightarrow \text{inst}(\text{sch}(P))
\]

\( K \) is a fixpoint of \( T_p \) if \( T_p(K) = K \)

Fact 3:

1. An instance \( K \) over \( \text{sch}(P) \) is a model of \( \Sigma_p \) iff \( T_p(K) \subseteq K \)
2. Each fixpoint of \( T_p \) is a model of \( \Sigma_p \).

A min of fixpoint of \( T_p \) containing \( I \) is a fixpoint of \( K \) of \( T_p \) containing \( I \) s.t. \( \forall \)fixpoint \( K' \) of \( T_p \) containing \( I \), \( K \subseteq K' \)

Fact 4: For every \( P \) and \( I \), \( T_p \) has a min fixpoint containing \( I \) and is equal \( P(I) \)

Proof: Goal is to show that \( T_p(P(I)) = P(I) \).

\( T_p(P(I)) \subseteq P(I) \) since \( P(I) \) is a normal of \( \Sigma_p \), by fact 3-2

Next. We show that \( P(I) \subseteq T_p(P(I)) \), \( T_p(P(I)) \subseteq P(I) \) and \( T_p \) is monotonic.

We have \( T_p(T_p(P(I))) \subseteq T_p(P(I)) \).

∴ By fact 3-2, we have that \( T_p(P(I)) \) is a modeled of \( \Sigma_p \).

∴ \( P(I) \) contains \( I \), \( T_p(P(I)) \) contains \( I \)

∴ \( T_p(P(I)) \) is a model of \( \Sigma_p \) that contains \( I \).

∴ \( P(I) \) is the min model among all models of \( \Sigma_p \) that contain \( I \), \( P(I) \subseteq T_p(P(I)) \)

\[
\text{TC}(x,y) :- \ E(x,y)
\]
\[
\text{TC}(x,z) :- \ E(x,y), \ \text{TC}(y,z)
\]

\( I = \{ E(1,2), E(3,4), E(2,3), E(4,6), E(8,7) \} \)
\[ Tp^1(I) = I \cup \{TC(1,2) \ldots TC(8,7)\} \]
\[ Tp^2(I) = Tp(Tp^1(I)) = I \cup \{TC(1,3), TC(3,4), TC(2,4), TC(1,2) \ldots TC(8,7)\} \]
\[ Tp^3(I) = Tp(Tp(Tp(I)))) = Tp^2(I) \cup \{TC(2,6), TC(1,4)\} \]
\[ Tp^4(I) = \ldots \cup \{TC(1,6)\} \]

Since \( I \subseteq Tp(I) \) (from definition)

We have \( I \subseteq Tp^1(I) \subseteq Tp^2(I) \subseteq Tp^3(I) \)

Let \( N \) be the No. of facts in \( B(P,I) \)

After \( \leq N \) steps, the consequence reaches a fixpoint, that is, \( Tp(Tp^N(I)) = Tp^w(I) \)

**Fact 5:** Let \( P \) be a datalog program and \( I \) an instance over the edbs of \( P \), then \( Tp^w(I) = P(I) \).

Proof: WTS, that \( Tp^w(I) \) is the min fixpoint of \( Tp \) containing \( I \) and from Fact 4, it follows that \( Tp^w(I) = P(I) \).

\[ Tp^w(I) = Tp^N(I). \]

\[ \therefore Tp(Tp^w(I)) = Tp(Tp^N(I)) = Tp^N(I) = Tp^w(I) \]

Next we show that \( Tp^w(I) \) is minimum.

Consider any \( J \) that is a fixpoint of \( Tp \) containing \( I \), \( I \subseteq J \).

**Base case.** \( Tp^0(I) = I \subseteq J. \)

Assume \( Tp^i(I) \subseteq J, \forall i \subseteq K \)

Ind. \( Tp(Tp^{k-1}(I)) = Tp(M) \) where \( M \subseteq J \)

Take a rule in \( P \), show that \( Tp(M) \subseteq J \).

Since \( J \) is a fixpoint, \( Tp^w(I) \) is the min fixpoint of \( Tp \) containing \( I \).

\[
\begin{align*}
S(X1, X3) &:\text{-} T(X1, X2), R(X2, a, X3) \\
T(X1, X4) &:\text{-} R(X1, a, X2), R(X2, b, X3), T(X3, X4) \\
T(X1, X3) &:\text{-} R(X1, a, X2), R(X2, a, X3)
\end{align*}
\]

Instance \( I \) \{ \( R(1,a,2), R(2,b,3), R(3,a,4), R(4,a,5), R(5,a,6) \) \}

![Datalog program and instance tree](image-url)