The minimax solution to the "Expert Setting"

Two-player zero-sum games:

1. Player A chooses an action $a_1$
2. Player B chooses an action $b_1$
3. Player A chooses an action $a_2$
   
4. Player B chooses an action $b_n$

Payoff to Player A: $g(a_1, b_1, a_2, \ldots, a_n, b_n)$}

Payoff to Player B: $-g(a_1, b_1, a_2, \ldots, a_n, b_n)$

i.e. we only need to worry about the payoff to one player.

2-player zero-sum games can be represented as a tree:
Examples of zero-sum games:

1. Heads up poker (ignoring the "rake")
2. Chess (payoff: +1 (win), 0 (draw), -1 (loss))
3. Sharing cake: "You split, I pick"

The minimax strategy in zero-sum games

- What if both players are playing optimally?
- What is the optimal strategy for each? What is the final payoff?

- For complex games (Chess), this can be difficult to compute → Deep Blue vs. Kasparov

- Not so bad for easy games. Let's consider the game of "You split, I pick." Here we can compute the "minimax" strategy as follows

\[ \begin{align*}
\text{Min} & \quad P_1, P_2 \in [0,1] \\
\text{Max} & \quad i \in 1,2 \\
& \quad P_i + P_2 = 1
\end{align*} \]

Player 1 chooses the split into \( P_1, P_2 \)
Player 2 chooses whatever piece he prefers
Player 2 gets the piece Pi
Modelling the "Expert Setting" as a 2-player 0-sum game.

Gambler vs Casino: On a sequence of rounds:

1. Gambler chooses a list \( \langle w_1, \ldots, w_n \rangle \) with \( \sum w_i = 1 \) over \( n \) "events".

2. Casino chooses the "outcome" of each event \( \langle l_1, \ldots, l_n \rangle \in \{0,1\}^n \).

\[ l_i = \begin{cases} 1 & \text{event } i \text{ "lost"} \\ 0 & \text{event } i \text{ "didn't lose"} \end{cases} \]

3. Gambler loses the bet's placed on lost events. That is, Gambler's loss is \( \sum_{i=1}^{n} w_i l_i = \mathbf{w} \cdot \mathbf{l} \).

Assume: this game stops once all events have lost more than \( R \) times, the "\( R \)-loss rule" i.e. "Gambler's not a complete fool" rule.

For the sake of the analysis, define the "state vector" \( \mathbf{S} = \langle S_1, \ldots, S_n \rangle \) as follows:

\[ S_i = \# \text{ of losses of event } i. \]

State vector starts at \( \mathbf{S} = \langle 0, 0, \ldots, 0 \rangle = 0^n \)
We can define the “Value of the game” to be the total loss of the Gambler when both players are playing optimally. More precisely, let \( V(S) \) be the total loss of the Gambler starting at \( S \).

We get a minimax expression for \( V(\cdot) \) as follows:

\[
V(S) = \begin{cases} 
0, & S_i \geq K \; \forall i = 1, \ldots, n \\
\min_{\hat{\omega}} \max_{\tilde{\omega} \in \gamma_0, 13} \tilde{\omega} \cdot \hat{\omega} + V(S+1), & \text{otherwise}
\end{cases}
\]

Some observations:

1. An optimal Gambler will never put weight \( w_i > 0 \) on an event \( i \) with \( S_i > K \).
2. Thus, the Casino has no incentive to inflict loss on such events.

With these observations in mind, we can actually solve the \( k=0 \) case easily.
\[
\begin{align*}
\text{h=1: } V(O^2) & = V(O^1) = \min_{w_1=1} \max_{l \in \mathbb{E}_2^{0,1,2}} w_1 l + V(\{1\}) \\
& = 1 + V(\{1\}) = 1
\end{align*}
\]

\[
\begin{align*}
\text{h=2: } V(O^3) & = \min_{w_1, w_2 \in \mathbb{E}_2^{0,1,2}} \max_{l_1, l_2 \in \mathbb{E}_2^{0,1,2}} w_1 l_1 + w_2 l_2 + V(\{l_1, l_2\}) \\
& = \max_{l_1, l_2} \left( \frac{l_1 + l_2}{2} + V(\{l_1, l_2\}) \right) \\
& = \max \left\{ \frac{1}{2} + V(\{0, 1\}), 1 + V(\{1, 2\}) \right\} \\
& = \max \left\{ \frac{3}{2}, \frac{3}{2} \right\} = \frac{3}{2}
\end{align*}
\]

\[
\begin{align*}
\text{h=3: } V(O^4) & = \min_{\mathbf{w} \in \mathbb{E}_3^{0,1,2}} \max_{l \in \mathbb{E}_3^{0,1,2}} w \cdot \hat{l} + V(\{l\}) \\
& = \max_{m \in \mathbb{E}_3^{0,1,2}} \frac{m}{3} + V(\{0\}) \\
& = \max \left\{ \frac{1}{3} + V(O^3), \frac{2}{3} + V(O^1) \right\} \\
& = \max \left\{ \frac{1}{3} + \frac{3}{2}, \frac{2}{3} + \frac{1}{2} \right\} \\
& = \frac{5}{6}
\end{align*}
\]
This analysis gives the recursion:

\[ V(O^n) = \frac{1}{n} + V(O^{n-1}) = \frac{1}{n} + \frac{1}{n-1} + V(O^{n-2}) \]

\[ = \sum_{i=1}^{n} \frac{1}{i} = H(n) \text{ the "harmonic series"} \]

Crucial observation: optimal play by the casino is in unit loss, i.e. only one event loses on each round.

For now, simply assume that this is the casino's play. Define,

\[ \hat{V}(\vec{s}) = \min \max_{\vec{w}} \sum_{i=1}^{n} \sum_{wi=1} \hat{V}(\vec{s} + \vec{e}_i) \]

We will show later that \( \hat{V}(\vec{s}) = V(\vec{s}) \).

To analyze \( V(\vec{s}) \), we consider a "random process" on the state vector \( \vec{s} \).
Markov Process: On each round, randomly pick a coordinate and increment \( \leq k \), otherwise do nothing.
Survival Probabilities: At a state $s$, what is the probability the last coordinate to "survive" is $i$? Call this $\hat{p}(s)$. 

\[ \hat{p}_2((0,0)) = \text{Prob that you start here and end up here} \]

\[ \hat{p}_1((1,0)) = \text{Prob that you start here} \]
Survival Probabilities, Part II
Expected Path Length: Define $\tau(T)$ to be the expected number of steps to reach the dead state.

$\tau = 5\frac{1}{2}$

$\tau = 4\frac{1}{2}$

$\tau = 4\frac{1}{2}$
Expected Path Length + Survival Probabilities

Notice a pattern???