WHY LOG PROBABILITIES?
CODING THEORY
RELATIVE ENTROPY

WANT TO SEND SYMBOL X ON CHANNEL

\[ X \quad \mathbb{P}(X=x_i) \quad -\log \mathbb{P}(x_i) \]

\[ \begin{array}{c|c|c}
  x_1 & 1/2 & 1 \\
  x_2 & 1/4 & 2 \\
  x_3 & 1/8 & 3 \\
  x_4 & 1/8 & 3 \\
\end{array} \] \{ \text{BITS} \}

MEASURE OF SURPRISE

\[ -\log 1 = 0 \quad \text{NO SURPRISE} \]

\[ -\log 0 = \infty \quad \text{INFINITE} \]

\[ -\log 2^i = i \quad \text{BITS} \]
Entropy equals expected surprise

\[ H(X) = \sum \ p(x_i) \log_2 \ \frac{1}{p(x_i)} \]

\[ = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 \]

\[ = \frac{3}{4} \]

Huffman code

| \( x_1 \) | \( \frac{1}{2} \) |
| \( x_2 \) | \( \frac{1}{4} \) |
| \( x_3 \) | \( \frac{1}{8} \) |
| \( x_4 \) | \( \frac{1}{8} \) |

Loop

Pick smallest two
Combine both into one
Sum their probs

Entropy = expected code length

\[ H(X) = \log_2 3 = 1.58 \text{ bits} \]

Expected code length

\[ \frac{1}{2} \left( 1 + 2 + 2 \right) \]

\[ = \frac{5}{2} \]

\[ = 1.64 \text{ bits} \]
**Code:** Assigns symbols a bitstring (codeword)
- Any sequence of codewords must be uniquely decodable

\[ L(C) = \sum_i p(x_i) l_c(x_i) \quad \uparrow \text{Expected codeword length} \]
\[ \text{Code} \quad \uparrow \text{Length of codeword for } x_i \]

**Optimal Code** \( C^* \)
- Minimum \( L(C) \)

**Thm:** \( H(X) \leq L(C^*) \leq H(X) + 1 \)

**Thm:** Huffman Codes are Optimal

**More Info:**
First five chapters of Cover & Thomas
RELATIVE ENTROPY

PROBABILITY VECTORS

\[ \Delta (p, q) = \sum p_i \ln \frac{p_i}{q_i} \]

SYMBOL USED FOR

DIVERGENCES

\[ \sum p_i \ln \frac{1}{q_i} - \sum p_i \ln \frac{1}{p_i} \]

EXPECTED CODELENGTH

OF BEST CODEBOOK

FOR \( q \)

EXPECTED CODE LENGTH

OF BEST CODEBOOK

FOR \( \bar{p} \)

BOTH EXPECTATIONS ARE WRT \( \bar{p} \)

\[ \Delta (\bar{p}, \frac{1}{n}) = \sum p_i \ln \frac{p_i}{\frac{1}{n}} \]

\[ = \sum p_i \ln p_i + \sum p_i \ln n \]

\[ = \ln n - H(\bar{p}) \]

\[ \geq 0 \]

= 0 AT CORNERS OF SIMPLEX

MAPLE PLOTS

\[ \Delta (\bar{p}, q) \]

NOT TOO STEEP AT BOUNDARY

\[ \Delta (\bar{p}, \bar{q}) \]

VERY STEEP

BARRIERS

FOR SIMPLEX