CMPS 201
Midterm 1
Review Problems

1. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$ and $f(n)$ is a polynomial satisfying $\text{deg}(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

2. The $n$th harmonic number is defined to be $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$. Use induction to prove that
   \[ \sum_{k=1}^{n} H_k = (n + 1)H_n - n \]
   for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$.)

3. Define the sequence $S_n$ by the recurrence $S_n = (n - 1) + \frac{n-1}{n^2} \cdot \sum_{k=1}^{n-1} S_k$. Use induction to prove $S_n \leq 2n$ for all $n \geq 1$.

4. The following sorting algorithm, called BadSort() is a modified version of StoogeSort() from the 2nd edition of CLRS, which seems to have been left out of the 3rd edition.

   ```
   BadSort(A, p, r)
   pre: p ≤ r
   3.   if p + 1 ≥ r
   4.       return
   5.   else
   6.       q = [(r - p + 1)/3]
   7.       BadSort(A, p, r - q)
   8.       BadSort(A, p + q, r)
   9.       BadSort(A, p, r - q)
   ```

   a. Use induction on the length $m = r - p + 1$ of $A[p \ldots r]$ to prove the correctness of BadSort().
   b. Write a recurrence relation for the number of array comparisons performed by BadSort() on an array of length $n$.
   c. Use the Master Theorem to find an asymptotic solution to this recurrence, and explain what is bad about BadSort().

5. Simplify the recurrence for MergeSort() by assuming that $n$ is an exact power of 2; $n = 2^k$ for some integer $k \geq 0$.

   \[ T(n) = \begin{cases} 
   0 & n = 1 \\
   2T \left( \frac{n}{2} \right) + (n - 1) & n \geq 2, n = 2^k
   \end{cases} \]

   Use the iteration method to find an exact solution to this recurrence.
6. Write a recursive algorithm (modeled on MergeSort()) that determines if an array is sorted, i.e. given an array \( A = (A_1, A_2, \ldots, A_n) \) as input, return TRUE/FALSE iff \( A \) is/is-not arranged in increasing order. Prove the correctness of your algorithm. Write a recurrence for the number \( T(n) \) of array comparisons performed by your algorithm. Check that \( T(n) = n - 1 \) is the exact solution to your recurrence.

7. Given \( A = (A_1, A_2, \ldots, A_n) \), a pair of indices \((i, j)\) is called an inversion iff both \( i < j \) and \( A_i > A_j \). Write a recursive algorithm that determines the number of inversions in its input array \( A \). Do this in such a way that the worst case number of comparisons performed is \( T(n) = \Theta(n \log n) \). (Hint: modify MergeSort() so that it counts inversions as it sorts.)