CMPS 201: Summer 2003
Homework Assignment 1

1. (10 Points) Prove that if \( f(n) = \Theta(g(n)) \), then \( f(n)^2 = \Theta(g(n)^2) \).

2. (10 Points) Let \( f(n) \) be a positive increasing function and \( c \) a positive constant. Is it necessarily true that \( f(cn) = \Theta(f(n)) \)? Either prove this or give a counter-example.

3. (10 Points) Let \( f(n) \) and \( g(n) \) be asymptotically positive functions. Prove that
   \[ f(n) + g(n) = \Theta(\max(f(n), g(n))) \]

4. (20 Points) List the following functions from lowest to highest asymptotic order. Indicate whether any two (or more) are of the same asymptotic order. Justify your answers.
   \[
   \begin{align*}
   2^\log(n) & \quad (\log n)^\log n & \quad n^{\log n} & \quad \ln(\ln(n)) & \quad 4^{\log n} & \quad \sqrt{2}^{\log n} & \quad n & \quad \sqrt{2n} & \quad 1 \\
   n^2 & \quad n! & \quad (3/2)^n & \quad n^2 & \quad (\log n)^2 & \quad 2^n & \quad n^2 & \quad n^{\log(\log n)} & \quad \ln(n) & \quad e^n \\
   2^{\sqrt{\log(n)}} & \quad 2^n & \quad n\log(n) & \quad 2^n & \quad \log(n) & \quad \sqrt{\log(n)} & \quad 2^{(\log n)^2} & \quad (n+1)! \\
   \end{align*}
   \]

5. (20 Points) Let \( f(n) \) and \( g(n) \) be positive functions. Prove or disprove each of the following.
   a. (10 Points) If \( f(n) = \Theta(g(n)) \) then \( 2^{f(n)} = \Theta(2^{g(n)}) \).
   b. (10 Points) If \( f(n) = \Theta(g(n)) \) then \( \log(f(n)) = \Theta(\log(g(n))) \). Assume here that \( f(n) \geq 2 \) and \( g(n) \geq 2 \) for all sufficiently large \( n \).

6. (20 Points) Determine the asymptotic order of each of the following expressions, i.e. for each expression, find a simple function \( g(n) \) such that the expression is in the class \( \Theta(g(n)) \). Prove your answers.
   a. (10 Points) \( \sum_{i=1}^{n} \log(i) \)
   b. (10 Points) \( \sum_{i=1}^{n} a^i \) where \( a > 0 \) is a constant. (Hint: consider the cases \( 0 < a < 1 \), \( a = 1 \), and \( a > 1 \) separately.)