Asymptotic Growth of Functions

We introduce several types of asymptotic notation which are used to compare the performance and efficiency of algorithms. As we’ll see, the asymptotic run time of an algorithm gives a simple, and machine independent, characterization of it’s complexity.

**Definition** Let \( g(n) \) be a function. The set \( O(g(n)) \) is defined as

\[
O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \}.
\]

In other words, \( f(n) \in O(g(n)) \) if and only if there exist positive constants \( c \), and \( n_0 \), such that for all \( n \geq n_0 \), the inequality \( 0 \leq f(n) \leq cg(n) \) is satisfied. We say that \( f(n) \) is Big O of \( g(n) \), or that \( g(n) \) is an asymptotic upper bound for \( f(n) \).

We often abuse notation slightly by writing \( f(n) = O(g(n)) \) to mean \( f(n) \in O(g(n)) \). Actually \( f(n) \in O(g(n)) \) is also an abuse of notation. We should really write \( f \in O(g) \) since what we have defined is a set of functions, not a set of numbers. The notational convention \( O(g(n)) \) is useful since it allows us to refer to the set \( O(n^3) \) say, without having to introduce a function symbol for the polynomial \( n^3 \). Observe that if \( f(n) = O(g(n)) \) then \( f(n) \) is asymptotically non-negative, i.e. \( f(n) \) is non-negative for all sufficiently large \( n \), and likewise for \( g(n) \). We make the blanket assumption from now on that all functions under discussion are asymptotically non-negative.

In practice we will be concerned with integer valued functions of a (positive) integer \( n \) (\( g : \mathbb{Z}^+ \to \mathbb{Z}^+ \)). However, in what follows, it is useful to consider \( n \) to be a continuous real variable taking positive values and \( g \) to be real valued function (\( g : \mathbb{R}^+ \to \mathbb{R}^+ \)).

Geometrically \( f(n) = O(g(n)) \) says:

![Graph showing \( f(n) \) and \( cg(n) \).]
Example 40n + 100 = O(n^2 + 10n + 300). Observe that 0 ≤ 40n + 100 ≤ n^2 + 10n + 300 for all \( n ≥ 20 \), as can be easily verified. Thus we may take \( n_0 = 20 \) and \( c = 1 \) in the definition.

Note that in this example, any value of \( n_0 \) greater than 20 will also work, and likewise any value of \( c \) greater than 1 works. In general if there exist positive constants \( n_0 \) and \( c \) such that \( 0 ≤ f(n) ≤ cg(n) \) for all \( n ≥ n_0 \), then infinitely many such constants also exist. In order to prove that \( f(n) = O(g(n)) \) it is not necessary to find the smallest possible \( n_0 \) and \( c \) making the \( 0 ≤ f(n) ≤ cg(n) \) true. It is only necessary to show that at least one pair of such constants exist.

Generalizing the last example, we will show that \( an + b = O(cn^2 + dn + e) \) for any constants \( a-e \), and in fact \( p(n) = O(q(n)) \) whenever \( p(n) \) and \( q(n) \) are polynomials with \( \deg(p) ≤ \deg(q) \).

Definition Let \( g(n) \) be a function and define the set \( \Omega(g(n)) \) to be

\[
\Omega(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n ≥ n_0 : 0 ≤ cg(n) ≤ f(n) \}.
\]

We say \( f(n) \) is big \( \Omega \) of \( g(n) \), and that \( g(n) \) is an asymptotic lower bound for \( f(n) \). As before we write \( f(n) = \Omega(g(n)) \) to mean \( f(n) ∈ \Omega(g(n)) \). The geometric interpretation is:
Lemma \( f(n) = O(g(n)) \) if and only if \( g(n) = \Omega(f(n)) \).

Proof: If \( f(n) = O(g(n)) \) then there exist positive numbers \( c_1, n_1 \) such that \( 0 \leq f(n) \leq c_1 g(n) \) for all \( n \geq n_1 \). Let \( c_2 = 1/c_1 \) and \( n_2 = n_1 \). Then \( 0 \leq c_2 f(n) \leq g(n) \) for all \( n \geq n_2 \), proving \( g(n) = \Omega(f(n)) \). The converse is similar and we leave it to the reader.

Definition Let \( g(n) \) be a function and define the set \( \Theta(g(n)) = O(g(n)) \cap \Omega(g(n)) \). Equivalently

\[
\Theta(g(n)) = \{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}.
\]

We write \( f(n) = \Theta(g(n)) \) and say the \( g(n) \) is an asymptotically tight bound for \( f(n) \), or that \( f(n) \) and \( g(n) \) are asymptotically equivalent. We interpret this geometrically as:

![Graph](image)

Exercise Prove that if \( c \) is a positive constant, then \( cf(n) = \Theta(f(n)) \).

Exercise Prove that \( f(n) = \Theta(g(n)) \) if and only if \( g(n) = \Theta(f(n)) \).

Example Prove that \( \sqrt{n+10} = \Theta(\sqrt{n}) \).

Proof: According to the definition, we must find positive numbers \( c_1, c_2, n_0 \), such that the inequality \( 0 \leq c_1 \sqrt{n} \leq \sqrt{n+10} \leq c_2 \sqrt{n} \) holds for all \( n \geq n_0 \). Pick \( c_1 = 1 \), \( c_2 = \sqrt{2} \), and \( n_0 = 10 \). Then if \( n \geq n_0 \) we have:

\[
-10 \leq 0 \quad \text{and} \quad 10 \leq n
\]

\[
-10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n
\]

\[
-10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n
\]

\[
c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n,
\]

\[
c_1^2 n \leq n+10 = c_2^2 n,
\]

\[
c_1 \sqrt{n} \leq \sqrt{n+10} \leq c_2 \sqrt{n},
\]

as required.
The reader may find our choice of values for the constants \( c_1, c_2, n_0 \) in this example somewhat mysterious. Adequate values for these constants can usually be obtained by working backwards algebraically from the inequality to be proved. Notice that in this example there are many valid choices. For instance one checks easily that \( c_1 = \sqrt{1/2}, c_2 = \sqrt{3/2}, \) and \( n_0 = 20 \) work equally well.

**Exercise** Let \( a, b \) be real numbers with \( b > 0 \). Prove directly from the definition (as above) that \( (n+a)^b = \Theta(n^b) \). (In what follows we learn a much easier way to prove this.)

**Lemma** If \( f(n) \leq h(n) \) for all sufficiently large \( n \), and if \( h(n) = O(g(n)) \), then \( f(n) = O(g(n)) \).

**Proof:** The above hypotheses say that there exist positive numbers \( c \) and \( n_1 \) such that \( h(n) \leq cg(n) \) for all \( n \geq n_1 \). Also there exists \( n_2 \) such that \( 0 \leq f(n) \leq h(n) \) for all \( n \geq n_2 \). (Recall \( f(n) \) is assumed to be asymptotically non-negative.) Define \( n_0 = \max(n_1, n_2) \), so that if \( n \geq n_0 \) we have both \( n \geq n_1 \) and \( n \geq n_2 \). Thus \( n \geq n_0 \) implies \( 0 \leq f(n) \leq cg(n) \), and therefore \( f(n) = O(g(n)) \).

**Exercise** Prove that if \( h_1(n) \leq f(n) \leq h_2(n) \) for all sufficiently large \( n \), where \( h_1(n) = \Omega(g(n)) \) and \( h_2(n) = O(g(n)) \), then \( f(n) = \Theta(g(n)) \).

**Example** Let \( k \geq 1 \) be a fixed integer. Prove that \( \sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \).

**Proof:** Observe that \( \sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1}) \), and

\[
\sum_{i=1}^{n} i^k \geq \sum_{i=1}^{\lfloor n/2 \rfloor} i^k \geq \sum_{i=1}^{\lfloor n/2 \rfloor} (n/2)^k \geq \lfloor n/2 \rfloor \cdot (n/2)^k \geq (n/2)(n/2)^k = (1/2)^{k+1} n^{k+1} = \Omega(n^{k+1}).
\]

By the result of the preceding exercise, we conclude \( \sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \).

When asymptotic notation appears in a formula such as \( T(n) = 2T(n/2) + \Theta(n) \) we interpret \( \Theta(n) \) to stand for some anonymous function in the class \( \Theta(n) \). For example \( 3n^3 + 4n^2 - 2n + 1 = 3n^3 + \Theta(n^2) \). Here \( \Theta(n^2) \) stands for \( 4n^2 - 2n + 1 \), which belongs to the class \( \Theta(n^2) \).

The expression \( \sum_{i=1}^{n} \Theta(i) \) can be puzzling. On the surface it stands for \( \Theta(1) + \Theta(2) + \Theta(3) + \cdots + \Theta(n) \), which is meaningless since \( \Theta(\text{constant}) \) consists of all functions which are bounded above by some constant. We interpret \( \Theta(i) \) in this expression to stand for a single function \( f(i) \) in the class \( \Theta(i) \), evaluated at \( i = 1, 2, 3, \ldots, n \).

**Exercise** Prove that \( \sum_{i=1}^{n} \Theta(i) = \Theta(n^2) \). The left hand side stands for a single function \( f(i) \) summed for \( i = 1, 2, 3, \ldots, n \). By the previous exercise it is sufficient to show that \( h_1(n) \leq \sum_{i=1}^{n} f(i) \leq h_2(n) \) for all sufficiently large \( n \), where \( h_1(n) = \Omega(n^2) \) and \( h_2(n) = O(n^2) \).
**Definition** \( o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \} \). We say that \( g(n) \) is a strict Asymptotic upper bound for \( f(n) \) and write \( f(n) = o(g(n)) \) as before.

**Lemma** \( f(n) = o(g(n)) \) if and only if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

**Proof:** Observe that \( f(n) = o(g(n)) \) if and only if \( \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c \), which is the very definition of the limit statement \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

**Example** \( \lg(n) = o(n) \) since \( \lim_{n \to \infty} \frac{\lg(n)}{n} = 0 \). (Apply l’Hopital’s rule.)

**Example** \( n^k = o(b^n) \) for any \( k > 0 \) and \( b > 1 \) since \( \lim_{n \to \infty} \frac{n^k}{b^n} = 0 \). (Apply l’Hopital’s rule \( \left\lceil k \right\rceil \) times.) In other words, any exponential grows strictly faster than any polynomial.

By comparing definitions of \( o(g(n)) \) and \( O(g(n)) \) one sees immediately that \( o(g(n)) \subseteq O(g(n)) \). Also no function can belong to both \( o(g(n)) \) and \( \Omega(g(n)) \), as is easily verified (exercise). Thus \( o(g(n)) \cap \Omega(g(n)) = \emptyset \), and therefore \( o(g(n)) \subseteq O(g(n)) - \Theta(g(n)) \).

**Definition** \( \omega(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \} \). Here we say that \( g(n) \) is a strict asymptotic lower bound for \( f(n) \) and write \( f(n) = \omega(g(n)) \).

**Exercise** Prove that \( f(n) = \omega(g(n)) \) if and only if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \).

**Exercise** Prove \( \omega(g(n)) \cap O(g(n)) = \emptyset \), whence \( \omega(g(n)) \subseteq \Omega(g(n)) - \Theta(g(n)) \).

The following picture emerges:
Lemma If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$, where $0 \leq L < \infty$, then $f(n) = O(g(n))$.

Proof: The definition of the above limit is $\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$. Thus if we let $\varepsilon = 1$, there exists a positive number $n_0$ such that for all $n \geq n_0$:

$$\left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$\therefore -1 < \frac{f(n)}{g(n)} - L < 1$$

$$\therefore \frac{f(n)}{g(n)} < L + 1$$

$$\therefore f(n) < (L + 1) \cdot g(n).$$

Now take $c = L + 1$ in the definition of $O$, so that $f(n) = O(g(n))$ as claimed. // /

Lemma If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$, where $0 < L \leq \infty$, then $f(n) = \Omega(g(n))$.

Proof: The limit statement implies $\lim_{n \to \infty} g(n) = L'$, where $L' = 1/L$ and hence $0 \leq L' < \infty$. By the previous lemma $g(n) = O(f(n))$, and therefore $f(n) = \Omega(g(n))$. // /

Exercise Prove that if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$, where $0 < L < \infty$, then $f(n) = \Theta(g(n))$.

Although $o(g(n))$, $\omega(g(n))$, and a certain subset of $\Theta(g(n))$ are characterized by limits, the full sets $O(g(n))$, $\Omega(g(n))$, and $\Theta(g(n))$ have no such characterization as the following examples show.

Example A Let $g(n) = n$ and $f(n) = (1 + \sin(n)) \cdot n$.

Clearly $f(n) = O(g(n))$, but $\frac{f(n)}{g(n)} = 1 + \sin(n)$, whose limit does not exist. This example shows that the containment $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$ is in general strict since $f(n) \neq \Omega(g(n))$ (exercise). Therefore $f(n) \neq \Theta(g(n))$, so that $f(n) \in O(g(n)) - \Theta(g(n))$. But $f(n) \neq o(g(n))$ since the limit does not exist.
Example B Let \( g(n) = n \) and \( f(n) = (2 + \sin(n)) \cdot n \).

Since \( n \leq (2 + \sin(n)) \cdot n \leq 3n \) for all \( n \geq 0 \), we have \( f(n) = \Theta(g(n)) \), but \( \frac{f(n)}{g(n)} = 2 + \sin(n) \) whose limit does not exist.

**Exercise** Find functions \( f(n) \) and \( g(n) \) such that \( f(n) \in \Omega(g(n)) - \Theta(g(n)) \), but \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \) does not exist (even in the sense of being infinite), so that \( f(n) \neq \omega(g(n)) \).

The preceding limit theorems and counter-examples can be summarized in the following diagram. Here \( L \) denotes the limit \( L = \lim_{n \to \infty} \frac{f(n)}{g(n)} \), if it exists.

In spite of the above counter-examples, the preceding limit theorems are a very useful tool for establishing asymptotic comparisons between functions. For instance recall the earlier exercise to show \( (n + a)^b = \Theta(n^b) \) for real numbers \( a \), and \( b \) with \( b > 0 \). The result follows immediately from

\[
\lim_{n \to \infty} \frac{(n + a)^b}{n^b} = \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^b = 1^b = 1,
\]

since \( 0 < 1 < \infty \).
Exercise  Use limits to prove the following:

a. \( n \lg(n) = o(n^2) \) (here \( \lg(n) \) denotes the base 2 logarithm of \( n \)).

b. \( n^2 2^n = \omega(n^{10}) \).

c. If \( P(n) \) is a polynomial of degree \( k \geq 0 \), then \( P(n) = \Theta(n^k) \).

d. \( f(n) + o(f(n)) = \Theta(f(n)) \). (One can always disregard lower order terms)

e. \( (\log n)^k = o(n^\varepsilon) \) for any \( k > 0 \) and \( \varepsilon > 0 \). (Polynomials grow faster than logs.)

f. \( n^\varepsilon = o(b^n) \) for any \( \varepsilon > 0 \) and \( b > 1 \). (Exponentials grow faster than polynomials.)

There is an analogy between the asymptotic comparison of functions \( f(n) \) and \( g(n) \), and the comparison of real numbers \( x \) and \( y \).

\[
\begin{align*}
  f(n) &= O(g(n)) \quad \sim \quad x \leq y \\
  f(n) &= \Theta(g(n)) \quad \sim \quad x = y \\
  f(n) &= \Omega(g(n)) \quad \sim \quad x \geq y \\
  f(n) &= o(g(n)) \quad \sim \quad x < y \\
  f(n) &= \omega(g(n)) \quad \sim \quad x > y
\end{align*}
\]

Note however that this analogy is not exact since there exist pairs of functions which are not comparable, while any two real numbers are comparable. (See problem 3-2c, p.58.)