1. Let \( T(n) \) satisfy the recurrence \( T(n) = aT(n/b) + f(n) \), where \( a \geq 1 \), \( b > 1 \) and \( f(n) \) is a polynomial satisfying \( \deg(f) > \log_b(a) \). Prove that case (3) of the Master Theorem applies, and in particular, prove that the regularity condition necessarily holds.

**Proof:**
Let \( d = \deg(f) \) and replace \( f(n) \) by the asymptotically equivalent function \( n^d \). We compare the polynomials \( n^d \) and \( n^{\log_b(a)} \). Let \( \epsilon = d - \log_b(a) \), which is positive since \( d > \log_b(a) \). Therefore \( d = \log_b(a) + \epsilon \), and \( n^d = \Omega(n^d) = \Omega(n^{\log_b(a)+\epsilon}) \), verifying the first hypothesis of case (3).

Observe \( d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1 \). Pick any \( c \) in the range \( a/b^d \leq c < 1 \). Then for any \( n \geq 1 \), we have \( a(n/b)^d = (a/b^d)n^d \leq cn^d \), verifying the regularity condition. \( \blacksquare \)

2. The \( n \)th harmonic number is defined to be \( H_n = \sum_{k=1}^{n} \frac{1}{k} \). Use induction to prove that
\[
\sum_{k=1}^{n} H_k = (n + 1)H_n - n
\]
for all \( n \geq 1 \). (Hint: Use the fact that \( H_n = H_{n-1} + \frac{1}{n} \))

**Proof:**
I. If \( n = 1 \), then \( H_1 = 1 \) and \( \sum_{k=1}^{1} H_k = 1 = 2 - 1 = (1 + 1) \cdot 1 - 1 = (1 + 1)H_1 - 1 \), so the base case is satisfied.

II. Let \( n > 1 \) be chosen arbitrarily, and assume \( \sum_{k=1}^{n-1} H_k = ((n - 1) + 1)H_{n-1} - (n - 1) \). We must show that \( \sum_{k=1}^{n} H_k = (n + 1)H_n - n \). We have
\[
\sum_{k=1}^{n} H_k = \sum_{k=1}^{n-1} H_k + H_n
\]
\[
= ((n - 1) + 1)H_{n-1} - (n - 1) + H_n \quad \text{by the induction hypothesis}
\]
\[
= nH_{n-1} - n + 1 + H_n
\]
\[
= nH_n - nH_n + nH_{n-1} - n + 1 + H_n
\]
\[
= (n + 1)H_n - n + 1 - n(H_n - H_{n-1})
\]
\[
= (n + 1)H_n - n + 1 - n \left( \frac{1}{n} \right) \quad \text{by the definition of } H_n
\]
\[
= (n + 1)H_n - n,
\]
as required. It follows that \( \sum_{k=1}^{n} H_k = (n + 1)H_n - n \) for all \( n \geq 1 \). \( \blacksquare \)
3. Define the sequence $S_n$ by the recurrence $S_n = (n - 1) + \frac{n - 1}{n^2} \cdot \sum_{k=1}^{n-1} S_k$. Use induction to prove $S_n \leq 2n$ for all $n \geq 1$.

**Proof:**
I. Observe $S_1 = (1 - 1) + \frac{1-1}{1^2} \cdot (\text{empty sum}) = 0 \leq 2 = 2 \cdot 1$, establishing the base case.

II. Let $n > 1$, and assume for all $k$ in the range $1 \leq k < n$ that $S_k \leq 2k$. We must show that $S_n \leq 2n$. We have

$$S_n = (n - 1) + \frac{n - 1}{n^2} \cdot \sum_{k=1}^{n-1} S_k$$

$$\leq (n - 1) + \frac{n - 1}{n^2} \cdot \sum_{k=1}^{n-1} 2k \quad \text{by the induction hypothesis}$$

$$= (n - 1) + \frac{(n-1)^2}{n}$$

$$= (n - 1) \left(1 + \frac{n-1}{n}\right)$$

$$= (n - 1) \left(1 + 1 - \frac{1}{n}\right)$$

$$= (n - 1) \left(2 - \frac{1}{n}\right)$$

$$= 2n - 2 - 1 + \frac{1}{n}$$

$$= 2n - 3 + \frac{1}{n}$$

$$\leq 2n$$

as required. It follows that $S_n \leq 2n$ for all $n \geq 1$. ■

4. The following sorting algorithm, called BadSort() is a modified version of StoogeSort() from the 2nd edition of CLRS, which seems to have been left out of the 3rd edition.

\[
\text{BadSort}(A, p, r) \quad \text{pre: } p \leq r
\]

3. if $p + 1 \geq r$
4. return
5. else
6. $q = \lfloor (r - p + 1)/3 \rfloor$
7. BadSort($A, p, r - q$)
8. BadSort($A, p + q, r$)
9. BadSort($A, p, r - q$)
a. Use induction on the length \( m = r - p + 1 \) of \( A[p \cdots r] \) to prove the correctness of \( \text{BadSort}(\cdot) \).

**Proof:**

I. If \( m = 1 \), then \( p = r \) so the test on line (1) is false and that on line (3) is true, so the algorithm returns with no changes to the array. Indeed, an array of length 1 is already sorted and no changes are necessary. If \( m = 2 \), then \( p + 1 = r \). Lines (1) and (2) insure that \( A[p] \) and \( A[p + 1] \) are arranged in increasing order. The test on line (3) is true so the algorithm returns with no other action. The base cases are therefore satisfied.

II. Let \( m > 2 \), and assume that \( \text{BadSort}(\cdot) \) correctly sorts an subarray of length less than \( m \). We must show that if \( m = r - p + 1 \), then \( \text{BadSort}(A, p, r) \) correctly sorts \( A[p \cdots r] \). After placing \( A[p] \) and \( A[r] \) in increasing order, the test on line (3) will be false (since \( m > 2 \Rightarrow r - p + 1 > 2 \Rightarrow r > p + 1 \) so lines (6)-(9) will be executed. Line (6) sets \( q = \lfloor m/3 \rfloor \), and since \( m \geq 3 \) we have \( q \geq 1 \). Therefore

\[
\text{length}(A[p \cdots (r - q)]) = r - q - p + 1 = m - q < m
\]

and

\[
\text{length}(A[(p + q) \cdots r]) = r - p - q + 1 = m - q < m.
\]

By our induction hypothesis, the effect of the recursive calls on lines (7)-(9) is to correctly sort the corresponding subarrays. It remains to show that this sequence of calls has the effect of sorting the subarray \( A[p \cdots r] \). To simplify the discussion, we define \( X, Y \) and \( Z \) to be the subarrays

\[
\begin{align*}
X &= A[p \cdots (p + q - 1)] \\
Y &= A[(p + q) \cdots (r - q)] \\
Z &= A[(r - q + 1) \cdots r]
\end{align*}
\]

After line (7) is executed, the subarray \( A[p \cdots (r - q)] = (X, Y) \) is sorted. Thus every element in \( X \) is less than or equal to every element in \( Y \), which we signify by writing \( X \leq Y \). After line (8) is executed, \( A[(p + q) \cdots r] = (Y, Z) \) is sorted, whence \( Y \leq Z \). Also \( X \leq Z \) since any element that was in \( X \) before the sort, and which belongs in \( Z \), was placed in \( Y \) by line (7), then placed in \( Z \) by line (8). In other words, all elements that ultimately belong in \( Z \) are placed there by the time (8) is executed. However \( X \leq Y \) may no longer be true at this point since some element that was originally in \( Z \), and is now in \( Y \), may be smaller than some element of \( X \). After line (9), we again have \( X \leq Y \), so the subarrays \( X, Y \) and \( Z \) are sorted and \( X \leq Y \leq Z \). Therefore \( A[p \cdots r] = (X, Y, Z) \) is now sorted, as required.

b. Write a recurrence relation for the number of array comparisons performed by \( \text{BadSort}(\cdot) \) on an array of length \( n \).

**Solution:**

At the top level of the recurrence, the sub-arrays have length \( n - q = n - \lfloor n/3 \rfloor = \lfloor 2n/3 \rfloor \). The (best, worst and average case) run time \( T(n) \) of \( \text{BadSort}(\cdot) \) therefore satisfies the recurrence

\[
T(n) = \begin{cases} 
1 & 1 \leq n < 3 \\
3T(\lfloor 2n/3 \rfloor) + 1 & n \geq 3
\end{cases}
\]

\[\text{■}\]

c. Use the Master Theorem to find an asymptotic solution to this recurrence, and explain what is bad about \( \text{BadSort}(\cdot) \).
Solution:
Simplifying the above recurrence for the Master Theorem gives $T(n) = 3T\left(\frac{n}{3^{1/2}}\right) + 1$. We compare $1 = n^0$ to $n^{\log_{3/2}(3)}$. Observe $3 > 1 \Rightarrow \log_{3/2}(3) > 0$, so setting $\epsilon = \log_{3/2}(3)$, we have $1 = n^0 = O(n^0) = O(n^{\log_{3/2}(3)-\epsilon})$. Case (1) yields $T(n) = \Theta(n^{\log_{3/2}(3)})$.

The runtime of most other sorting algorithms is no worse than $\Theta(n^2)$. For instance MergeSort and HeapSort run in (worst case) $\Theta(n \log n)$ time, while InsertionSort and QuickSort run in time $\Theta(n^2)$ (again worst case). But $\log_{3/2}(3) = 2.7095\ldots$, so BadSort runs in $\Theta(n^{2.7095\ldots})$ time, which is considerably worse than any standard sorting algorithm, making it aptly named.

5. Simplify the recurrence for MergeSort() by assuming that $n$ is an exact power of 2; $n = 2^k$ for some integer $k \geq 0$.

$$T(n) = \begin{cases} 
0 & n = 1 \\
2T\left(\frac{n}{2}\right) + (n - 1) & n \geq 2, n = 2^k 
\end{cases}$$

Use the iteration method to find an exact solution to this recurrence.

Solution:
Recurring down to depth $k$ yields:

$$T(n) = (n - 1) + 2T\left(\frac{n}{2}\right)$$

$$= (n - 1) + 2\left[\left(\frac{n}{2} - 1\right) + 2T\left(\frac{n}{2^2}\right)\right]$$

$$= (n - 1) + (n - 2) + 2^2T\left(\frac{n}{2^2}\right)$$

$$= (n - 1) + (n - 2) + 2^2\left[\left(\frac{n}{2^2} - 1\right) + 2T\left(\frac{n}{2^3}\right)\right]$$

$$= (n - 1) + (n - 2) + (n - 2^2) + 2^3T\left(\frac{n}{2^3}\right)$$

$$\vdots$$

$$= \sum_{i=0}^{k-1} (n - 2^i) + 2^k T\left(\frac{n}{2^k}\right).$$

The recursion halts when $k$ satisfies: $\frac{n}{2^k} = 1 \Leftrightarrow n = 2^k \Leftrightarrow k = \lg n$. For this $k$ we have

$$T(n) = \sum_{i=0}^{k-1} (n - 2^i) + 2^k T(1)$$

$$= \sum_{i=0}^{k-1} n + \sum_{i=0}^{k-1} 2^i + 2^k \cdot 0$$
\[ k \cdot n + \frac{2^k - 1}{2 - 1} \]
\[ = n \lg n + 2^k - 1, \]
and hence \( T(n) = n \lg n + n - 1 \). One may also check directly that this function solves the above recurrence. First \( T(1) = 1 \cdot 0 + 1 - 1 = 0 \). For the recursive branch,

\[
RHS = 2T\left(\frac{n}{2}\right) + (n - 1)
\]
\[ = 2 \left[ \left(\frac{n}{2}\right) \lg \left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) - 1 \right] + (n - 1) \]
\[ = n \left(\lg n - \lg 2\right) + n - 2 + n - 1 \]
\[ = n \lg n - n \lg 2 + 2n - 1 \]
\[ = n \lg n - n + 1 \]

\[ \blacksquare \]