Games with continuous strategy spaces

Exhaustive search does not work

Example: price competition of identical products

2 firms
per unit cost: $10 for both

Suppose customers only buy from cheapest firm
- if both choose identical prices, they split sales evenly

\[ Q = 100 - p \]

\[ D_i (p_i, p_2) = \begin{cases} 
100 - p_i & p_i < p_2 \\
\frac{100-p_i}{2} & p_i = p_2 \\
0 & p_2 < p_i 
\end{cases} \]

\[ D_i (c_i, c_j) \text{ if } p_c = 40 \]

\[
\text{Demand graph with price on y-axis and units on x-axis.}
\]
Payoff of firm if itchose y1:

\[
\Pi_1(p_1, p_2) = \begin{cases} 
(p_1-10)(100-p_1) & p_1 < p_2 \\
(p_1-10)(100-p_1)/2 & p_1 = p_2 \\
0 & p_2 < p_1
\end{cases}
\]

Special case \( p_2 = 40 \)

To find an NE, we must find a pair of prices \((p_1, p_2)\) such that no player can do better by deviating.

- Any \((p_1, p_2)\) with \(\min(p_i) < 10\) is not a NE.
  Why? Because a firm would lose money selling below cost. If the firm chose \(p_1 = 100\), the firm could provide a payoff.

- Any \((p_1, p_2)\) with \(p_1, p_2 > 10\) \(p_1 > p_2\) is not a NE.
  Why? The firm with the lower price makes a profit, but this firm switched prices.
why? Firm 1 sets 0 profit if this profile. By chasing to $P_2 - E$ where $E$ arbitrarily small, he sets a positive profit.

- Any $(C_1, C_2)$ with $P_1, P_2 > 10$ and $P_2 > P_1$ is not NE. Why? Symmetry with last point.

- Any $(C_1, C_2)$ with $P_1 = P_2 = 10$ is not a NE. Fim 1's profit is $\frac{1}{2}(P - 10)(100 - P)$ by switching to price $P - E$, his payoff is 

$$\frac{1}{2}(P - E - 10)(100 - P - E)$$

For $E$ small, (2) is close to double (1) (both are positive).

- Is: $(P_1 = 10, P_2 = 10)$ a NE?

Both firms make 0 profit.

If firm 1 raises his price to $P_1 > P_1$, profit is still 0.

If firm 1 lowers his price to $P_1 < P_1$, his payoff becomes negative. Thus keeping $P_1 = 10$ is a B. R.

By symmetry same is true for player 2.
Price matching:

Each store promises to match each other's price.

\[
\left( \min(P_1, P_2) - 10 \right) \frac{1}{2} \left( 100 - \min(P_1, P_2) \right)
\]

If only one shop:

\[
(p - 10)(100 - p)
\]

\[
\Pi = -p^2 + 110p - 1000
\]

\[
\frac{d\Pi}{dp} = -2p + 110
\]

\[
\frac{d\Pi}{dp} (p^*) = 0 \rightarrow -2p + 110 = 0
\]

\[
p^* = 55
\]

Suppose shop 2 chooses \( P_2 < 55 \): what is its best response?

\[
\text{Best response of } 2 \text{ if } P_2 < 55
\]

What if \( P_2 \geq 55 \):

\[
\text{Best response of } 2 \text{ if } P_2 \geq 55
\]
What if $p_2 = 55$?

Is $(p_1, p_2)$ with $p_1 < p_2 \leq 55$ a NE?

No, Player 1 can increase profit by raising price to $p_1 = p_2$.

Is $(p_1, p_2)$ with $p_1 = p_2 \leq 55$ a NE?

Yes. No player changes payoff by raising price. Players lower payoff by lowering price.

Is $p_1, p_2$ with $55 < p_1$ and $55 < p_2$ a NE?

No. Low-priced player can improve payoff by lowering price to 55.