On the theory of fourth-order tensors and their applications in computational mechanics

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Dedicated to Professor Dr. -Ing. Yavuz Başar on the occasion of his 65th anniversary

Abstract

Many problems concerned with the mathematical treatment of fourth-order tensors still remain open in the literature. In the present paper they will be considered in the framework of a complete theory involving a set of notations and definitions, a tensor operation algebra, differentiation rules, eigenvalue problems, applications of fourth-order tensors to isotropic tensor functions and some other relevant aspects. A tensor is understood to be an invariant quantity with respect to any coordinate system transformation, which justifies the use of absolute notation preferred in this work. As a most important application field of fourth-order tensors, elastic moduli are formulated in a material and a spatial description and given for various hyperelastic material models. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Fourth-order tensors as a mathematical object have found in the last 20 years a wide use in computational mechanics and especially in the finite element method. Their well-known applications are tangent (elastic or elasto-plastic) moduli as well as damage tensors playing an important role in the formulation of constitutive and evolution equations. The use of fourth-order tensors remains, nevertheless, extremely complicated because of the absence of a closed theory embracing many relevant aspects. In spite of active efforts in recent times (see [3–9,11,21,22,24,27–30,37,38,41,42]), there are many problems concerned with the treatment of fourth-order tensors, which are not completely solved and still remain open in the literature.

• How can a fourth-order tensor be constructed from two second-order ones?
• How can algebraic operations with fourth-order tensors be given in the absolute notation?
• What does the derivative of a second-order with respect to another one mean? There is no unified definition for the derivative with respect to a tensor. Is there a difference in the definition of the derivative with respect to a symmetric second-order tensor?
• How can the eigenvalue problem be formulated for fourth-order tensors and how can principal invariants and eigenvalues of a fourth-order tensor be calculated? How many irreducible invariants does a fourth-order possess?
The goal of the present paper is to try to solve these problems in a framework of the complete theory and to demonstrate then its application for the derivation of tangent moduli. The important characteristic feature of the development is the absolute notation used for tensor quantities throughout the paper. This is not only a notation manner but the basic concept of the paper. A tensor is understood to be an invariant quantity with respect to any co-ordinate system transformation and can be neither covariant, nor contravariant, nor mixedvariant in contrast to its components. In this connection, for example, the widespread use of the metric tensor as an argument of a function or as a differentiation parameter seems to be meaningless, since the metric tensor represents in the end the identity tensor. Such notations always have to be accompanied by an indication of the component variance to be used in the present operation and will be avoided here.

As a further crucial point of the paper it should be noted that in contrast to many other works we deal with arbitrary non-symmetric fourth-order tensors and understand them as a linear mapping of one arbitrary second-order tensor into another one (see also [19,29]). This considerably simplifies the treatment of fourth-order tensors, which can be seen especially clearly in consideration of the eigenvalue problem. The symmetry properties of fourth-order tensors, which are relevant, for instance, for tangent moduli, can be enforced without loss of generality at the last stage of derivation.

The starting point of the development is the set of notations and definitions given in Section 2. To have more freedom in construction of fourth-order tensors we propose a new tensor product of two second-order ones. The crucial point of the presented theory lies in a new definition of the derivative of one second-order tensor with respect to another one. Due to this definition, the well-known product derivative rule valid for scalar values is proved to hold also for second-order tensors. The double contraction of a fourth-order tensor with a second-order one is also redefined to be consistent with the differentiation rule introduced, such that the important rate relation is satisfied. Further, it is shown that the derivative with respect to a symmetric tensor is not unique and requires a slight correction in the definition. Special attention is paid in Section 2 to the transposition operations for fourth-order tensors and the resultant definition of a symmetric and a super-symmetric tensor. In this context, we introduce a special transposition operation bringing a fourth-order tensor to the standard definition of the tensor derivative. In the following this will permit to verify some important results by comparing them with those available in the literature.

The indispensable part of the proposed theory is the algebra of fourth-order tensors given in Section 3. Simple, double, quadruple contractions and other operations with fourth-order tensors constructed from second-order ones are presented in absolute notation and can easily be proved using the definitions of the previous section. The following important topic is the formulation and proof of the differentiation rules such as the product derivative and the chain rule. On this basis, we then obtain the derivative of the power tensor function for positive as well as negative integer exponents.

Section 4 is devoted to the formulation of eigenvalue problems and spectral decomposition of fourth-order tensors. Using matrix representation the eigenvalue problem of a fourth-order tensor is reduced to that of a matrix and can then be solved by a standard procedure. For a symmetric fourth-order tensor this yields nine real eigenvalues and nine corresponding eigentensors. A complete analogy with the eigenvalue problem of a second-order tensor can be observed. On the basis of the other possible definitions for the double contraction at whole 27 principal invariants and 27 eigenvalues of the fourth-order tensor can be received in this way. The discussed procedure is illustrated by spectral decompositions for a number of important fourth-order tensors like the identity, the trace projection and the transposition projection tensor.

Application of the proposed theory to isotropic tensor functions is the topic of Section 5. Attention is focused on the representation of the derivative of an isotropic tensor-valued tensor function with respect to its argument, which is of major importance for the formulation and numerical calculation of tangent moduli in large strain elasticity and elasto-plasticity. We start with the consideration of a special class of isotropic tensor functions, which can be expanded in tensor power series. In the case of a symmetric tensor argument we receive for the derivative in question two closed-form representations. The first one is expressed in terms of eigenvectors of the tensor argument and corresponds to the well-known result (see [9,27]). The second representation is given through the eigenvalue bases and is advantageous, as it avoids the numerically expensive computation of eigenvectors. In contrast to the analogous result by Miehe [24],
this representation does not use the derivatives of the eigenvalue bases and is obtained in a simpler and more compact form.

The tensor algebra and differentiation rules developed are applied in Section 6 to formulate tangent moduli in a material and a spatial description. It is proved that the tangent moduli presented are symmetric fourth-order tensors and permit consequently the spectral decomposition. The important question to be treated then is how the isotropy can be identified by an elastic modulus. Using the spectral decomposition we prove that the elastic modulus corresponds to an isotropic material in the natural state if only it can be represented in the form of the St. Venant–Kirchhoff model involving two material parameters (see also [14,15,17,19,20,29,31] and references therein). Elastic moduli for this and some other hyperelastic material models such as Mooney–Rivlin and Ogden are given in Section 7.

2. Basic notations and definitions

A second-order tensor (a bold capital letter):
\[ A = A_{ij} g^i \otimes g^j = A_{ij}^k g_i \otimes g^j = \cdots \]  (2.1)

A fourth-order tensor (a bold italics capital letter):
\[ D = D_{ijkl} g^i \otimes g^j \otimes g^k \otimes g^l = D_{ijkl}^k g_i \otimes g^j \otimes g^k \otimes g^l = \cdots \]  (2.2)

The simple contraction of arbitrary order tensors:
\[ AD = (A_{ij} g_i \otimes g^j)(D^{km}_{ij} g_i \otimes g^i \otimes g^k \otimes g^m) = A_{ij}^{'} D^{km}_{ij} g_i \otimes g^i \otimes g^k \otimes g^m. \]  (2.3)

Tensor products of two second-order tensors are defined as follows:
\[ D = A \otimes B = (A_{ij}^k g_i \otimes g^j \otimes g^k \otimes g^l) = A_{ij}^l B_{ij}^k g_i \otimes g^k \otimes g^l \]  (2.4)
\[ D = A \times B = (A_{ij}^l g_i \otimes g^j \otimes g^j \otimes g^l) = A_{ij}^l B_{ij}^k g_i \otimes g^k \otimes g^j \otimes g^l. \]  (2.5)

It can easily be confirmed the tensor products satisfy the distributive rule:
\[ A \otimes (B + C) = A \otimes B + A \otimes C, \quad A \times (B + C) = A \times B + A \times C. \]  (2.6)

Transposition operations (\(\cdots\)^T and \(\cdots\)^t) for fourth-order tensors:
\[ D^T = (D_{ij}^{jk} g_i \otimes g^j \otimes g^k \otimes g^l)^T = D_{ij}^{kj} g^i \otimes g^j \otimes g^k \otimes g^l, \]  (2.7)
\[ D^t = (D_{ij}^{jk} g_i \otimes g^j \otimes g^k \otimes g^l)^t = D_{ij}^{kj} g_i \otimes g^j \otimes g^k \otimes g^l. \]  (2.8)

**Remark.** One can easily prove that the transposition operations (\(\cdots\)^T and \(\cdots\)^t) are not commutative with each other: \(D^T \neq D^t\).

According to the above definitions the transposition applied to the tensor products (2.4) and (2.5) yields:
\[ (A \otimes B)^T = A^T \otimes B^T, \quad (A \times B)^T = B \times A, \quad (A \times B)^t = A \times B^t. \]  (2.9)

A symmetric fourth-order tensor (cf. [19,29])
\[ D^T = D. \]  (2.10)

A super-symmetric fourth-order tensor
\[ E^T = E \quad \text{and} \quad E^t = E. \]  (2.11)
By means of the operation \((\cdots)')\)

\[ E' = \frac{1}{2}(E + E'), \tag{2.12} \]
a symmetric fourth-order tensor can be transformed into a super-symmetric one.

**Remark.** For a fourth-order tensor some other transposition operations can be defined and particularly:

\[(\cdots)^R : D^R = (D^{jk}_j g_i \otimes g'_i \otimes g_{ik} \otimes g'_k)^R = D^{ik}_j g_i \otimes g'_j \otimes g'_i \otimes g_k, \tag{2.13} \]

which possess the important property

\[(A \times B)^R = A \otimes B. \tag{2.14} \]

Double contractions of a fourth-order tensor with a second-order one \((4 : 2)\) and \((2 : 4)\) are defined by

\[ D : A = (D^{jk}_j g_i \otimes g'_i \otimes g_{ik} \otimes g'_k) : (A^m_{nsm} \otimes g^n) = D^{jk}_j A'_{ik} g_i \otimes g'_k, \tag{2.15} \]

\[ A : D = (A^m_{nsm} \otimes g^n) : (D^{jk}_j g_i \otimes g'_i \otimes g_{ik} \otimes g'_k) = A'_{ik} D^{jk}_j g_i \otimes g'_k. \tag{2.16} \]

The double contraction of a fourth-order tensor with another fourth-order one \((4:4)\):

\[ D : A = (D^{jk}_j g_i \otimes g'_i \otimes g_{ik} \otimes g'_k) : (A'^m_{nrms} \otimes g^n \otimes g'_r \otimes g'_l) = D^{jk}_j A'^r_{ik} g_i \otimes g'_k \otimes g'_r \otimes g'_l. \tag{2.17} \]

Evidently, the double contraction of fourth-order tensors is not commutative \((D : A \neq A : D)\), but can easily be proved to fulfil the associative rule:

\[(A : B) : C = A : (B : C). \tag{2.18} \]

In contrast to the double contraction the scalar product (quadruple contraction) of two fourth-order tensors \((4 : 4)\)

\[ D :: A = (D^{jk}_j g_i \otimes g'_i \otimes g_{ik} \otimes g'_k) :: (A'^m_{nrms} \otimes g^n \otimes g'_r \otimes g'_l) = D^{ik}_j A'^r_{jk} \tag{2.19} \]

obey the commutative rule:

\[ D :: A = A :: D. \tag{2.20} \]

According to the definitions \((2.15)\)–\((2.17)\) and \((2.19)\) the operations \((2 : 4)\), \((4 : 2)\), \((4 : 4)\) and \((4 :: 4)\) satisfy the distributive rule e.g.:

\[ A : (B + C) = A : B + A : C, \quad (A + B) : C = A : C + B : C, \tag{2.21} \]

\[ A : (B + C) = A : B + A : C, \quad (A + B) : C = A : C + B : C, \tag{2.22} \]

\[ A :: (B + C) = A :: B + A :: C, \quad (A + B) :: C = A :: C + B :: C. \tag{2.23} \]

The second-order identity tensor:

\[ I = g_i \otimes g'^i. \tag{2.24} \]

The fourth-order identity tensor:

\[ I = I \otimes I = g_i \otimes g'^i \otimes g_j \otimes g'^j. \tag{2.25} \]
The trace of a fourth-order tensor is defined by
\[ \operatorname{tr} D = D : I = I : D = D_{ij}^j. \]  
(2.26)

Powers of fourth-order tensors:
\[ D^0 = I, \quad D^1 = D, \quad D^2 = D : D, \ldots \quad D^n = D : D : \cdots : D. \]  
(2.27)

An inverse fourth-order tensor:
\[ D : D^{-1} = D^{-1} : D = I. \]  
(2.28)

The matrix representation of a fourth-order tensor (\(\ldots\)) is defined through its mixedvariant components by
\[
D = \begin{bmatrix}
D_{11}^{11} & D_{12}^{11} & D_{13}^{11} & D_{14}^{11} \\
D_{11}^{12} & D_{12}^{12} & D_{13}^{12} & D_{14}^{12} \\
D_{11}^{13} & D_{12}^{13} & D_{13}^{13} & D_{14}^{13} \\
D_{11}^{14} & D_{12}^{14} & D_{13}^{14} & D_{14}^{14}
\end{bmatrix}.
\]  
(2.29)

The partial derivative of a scalar-valued tensor function \(z(A)\) with respect to its second-order argument \(A_{(0,2)}\) is given by
\[
z(A)_{A_{ij}} = \frac{\partial z}{\partial (A_{ij})} = \frac{\partial z}{\partial A_{ij}} g^i \otimes g^j
\]  
(2.30)

and possesses the following important properties:

1. The derivative is independent of the choice of the component variance used in the differentiation. Since \(A^k = A^m_{\mu} g^\mu\), we have
\[ \frac{\partial z}{\partial A_{ij}} g^i \otimes g^j = \frac{\partial z}{\partial A^k}_{ij} g^i \otimes g^j = \frac{\partial z}{\partial A^k} g^i \otimes g^j = \frac{\partial z}{\partial A^k} g^i \otimes g^j. \]

Similarly it can easily be proved also for the covariant components.

2. The derivative is independent of the choice of the co-ordinate system. In a transformed co-ordinate system we obtain:
\[
A = A_{ji} g^i \otimes g^j = \bar{A}_{ji} g^i \otimes g^j \Rightarrow \bar{A}^k_i = (g^k \cdot g_i) A_{ji}(g^j \cdot g_j),
\]
\[ \frac{\partial z}{\partial A_{ij}} g^i \otimes g^j = \frac{\partial z}{\partial A^k_{ij}} g^i \otimes g^j = \frac{\partial z}{\partial \bar{A}^k_{ij}} (g^k \cdot g_i)(g^j \cdot g_j) g^i \otimes g^j = \frac{\partial z}{\partial \bar{A}^k_{ij}} g^i \otimes g^j. \]

These properties ensure the objectivity of the tensor derivative and justify the use of the absolute notation \(z(A)_{A}\).

**Remark 1.** Properties 1 and 2 and their proofs are not valid for the derivative with respect to the metric tensor. As mentioned above, the derivative with respect to the metric tensor is not objective since it depends not only on the component variance but also on the co-ordinate system used in differentiation (see [16]).

**Remark 2.** It is important to note that in the differentiation with respect to a tensor its vector base is kept constant. Disregard of this fact may lead to an incorrect result. As a typical example we consider derivatives with respect to the Green–Lagrange and Almansi strain tensors.
\[ E = \frac{1}{2} (C - G) = \frac{1}{2} (g_{ij} - G_{ij}) \mathbf{G}^i \otimes \mathbf{G}^j = E_{ij} \mathbf{G}^i \otimes \mathbf{G}^j \]

and

\[ e = \frac{1}{2} (g - b^{-1}) = \frac{1}{2} (g_{ij} - G_{ij}) \mathbf{g}^i \otimes \mathbf{g}^j = e_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \]

respectively, which are often used for the formulation of constitutive relations. It can be seen that these tensors have equal covariant components \( E_{ij} = e_{ij} \) related, nevertheless, to the different bases remaining constant in the differentiation. Thus

\[
 F = \frac{\partial \psi}{\partial E} F^T = F \left( \frac{\partial \psi}{\partial g_{ij}} \mathbf{G}_i \otimes \mathbf{G}_j \right) F^T = F \left( 2 \frac{\partial \psi}{\partial g_{ij}} \mathbf{g}_i \otimes \mathbf{g}_j \right)^T \frac{\partial \psi}{\partial e} \mathbf{g}_i \otimes \mathbf{g}_j,
\]

which evidently contradicts the constitutive equation for the Cauchy stress tensor of the form \( \tau = \psi_e \) (see e.g. [32]).

The partial derivative of a second-order tensor \( \mathbf{C} = \mathbf{C}_j^i \mathbf{G}_i \otimes \mathbf{G}_j \) with respect to another second-order one \( \mathbf{A} = \mathbf{A}_k^l \mathbf{g}_k \otimes \mathbf{g}_l \) (2.32) is defined by

\[
 C_{iA} = \frac{\partial (\mathbf{C}_j^i \mathbf{G}_i \otimes \mathbf{G}_j)}{\partial (\mathbf{A}_k^l \mathbf{g}_k \otimes \mathbf{g}_l)} = \frac{\partial \mathbf{C}_j^i}{\partial \mathbf{A}_k^i} \mathbf{G}_i \otimes \mathbf{g}_k \otimes \mathbf{g}_l \otimes \mathbf{G}_j
\]

for the tensor bases \( \mathbf{G}_i \otimes \mathbf{G}_j \) of \( \mathbf{C} \), which are independent of the tensor components \( \mathbf{A}_k^l \). In other respect the choice of co-ordinate systems and the variance of components of both tensors \( \mathbf{A} \) and \( \mathbf{C} \) are irrelevant, which can easily be proved similarly to the properties of the derivative (0,2). This ensures the objectivity of the definition (2.31) and justifies the use of absolute notation for the tensor \( C_{iA} \).

Remark. The derivative obtained according to the definition (2.31) can be transformed to the standard one by means of transposition \((\cdots)^T\). Using (2.13) we have:

\[
 (C_{iA})^T = \frac{\partial C^i_j}{\partial A_k^i} \mathbf{G}_k \otimes \mathbf{G}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_i.
\]

The definitions (2.30) and (2.31) become non-unique if the tensor \( \mathbf{A} \) is symmetric. In this case the differentiation cannot still be carried out with respect to nine components of \( \mathbf{A} \), since only six of them are independent. This deficiency can be eliminated by modifying the definitions (2.30) and (2.31). Accordingly the derivatives with respect to the symmetric tensor \( \mathbf{A} = \mathbf{A}^T \) can be defined by

\[
 z(\mathbf{A})_{,A} = \frac{1}{2} \sum_{i,j \leq l} \frac{\partial z}{\partial A_{ij}} (\mathbf{g}_i \otimes \mathbf{g}_j + \mathbf{g}_j \otimes \mathbf{g}_i) = \frac{1}{2} \sum_{i,j \leq l} \frac{\partial z}{\partial A^{ij}} (\mathbf{g}^i \otimes \mathbf{g}^j + \mathbf{g}^j \otimes \mathbf{g}^i)
\]

\[
 = \frac{1}{2} \sum_{i,j \leq l} \frac{\partial z}{\partial A^{ij}} (\mathbf{g}^i \otimes \mathbf{g}_j + \mathbf{g}_j \otimes \mathbf{g}^i),
\]

\[
 C_{iA} = \frac{1}{2} \sum_{k,l \leq k} \frac{\partial C^j_i}{\partial A^{kl}} \mathbf{G}_k \otimes (\mathbf{g}_k \otimes \mathbf{g}_l + \mathbf{g}_l \otimes \mathbf{g}_k) \otimes \mathbf{G}_j = \frac{1}{2} \sum_{k,l \leq k} \frac{\partial C^j_i}{\partial A^{kl}} \mathbf{G}_k \otimes (\mathbf{g}^k \otimes \mathbf{g}^l + \mathbf{g}^l \otimes \mathbf{g}^k) \otimes \mathbf{G}_j
\]

\[
 = \frac{1}{2} \sum_{k,l \leq k} \frac{\partial C^j_i}{\partial A^{kl}} \mathbf{G}_k \otimes (\mathbf{g}^k \otimes \mathbf{g}_l + \mathbf{g}_l \otimes \mathbf{g}^k) \otimes \mathbf{G}_j.
\]

As can be observed the differentiation in (2.32) and (2.33) is accomplished with respect to the six independent components of \( \mathbf{A} \) and the resulting tensor is then symmetrized.
3. The algebra of fourth-order tensors

In the following we summarize some important tensor identities, which will be intensively used in further derivation:

\[
\begin{align*}
A \otimes B & : C = ACB, \\
(A \times B) & : C = (B : C)A, \\
(A \otimes B)C & = A \otimes (BC), \\
(A \otimes B)C & = (AC) \times B, \\
(A \otimes B) : (C \otimes D) & = (AC) \otimes (DB), \\
(A \otimes B) : (C \times D) & = (ACB) \times D, \\
(A \otimes B) : (C \times D) & = (ACB) \times D, \\
(A \otimes B)^T : (C \otimes D) & = (AD^T) \otimes (C^T B)^T, \\
A \otimes B & : (C \otimes D) = (AC) \otimes (DB), \\
(A \otimes B) : (C \otimes D) & = (A \otimes B) : (C \otimes D) = A \otimes B - (A \otimes B)^T : (C \otimes D) = A \otimes B - C^T BD^T, \\
(A \otimes B) & : (C \times D) = (AC) \otimes (DB), \\
(A \otimes B) & : (C \times D) = (AC) \otimes (DB), \\
A \otimes B & : (C \times D) = (AC) \otimes (DB), \\
(A \otimes B)^T & : (C \otimes D) = (AC)^T B \times D, \\
(E : F)^T & = E^T : F, \\
E^T : A & = A : E, \\
A(E : C)B & = (AEB) : C.
\end{align*}
\]

The above relations directly emanate from the corresponding definitions (2.3)–(2.5), (2.7)–(2.8), (2.15)–(2.17) and (2.19). For example the identity (3.5), can easily be proved using (2.17), which leads to

\[
(A \otimes B) : (C \otimes D) = (A^T B^T)_{ij} g_i \otimes g_j \otimes g_i \otimes g_j = (C^T D)_{ij} g_i \otimes g_j \otimes g_i \otimes g_j = (AC) \otimes (DB).
\]

By virtue of (2.31) we construct the partial derivative of a second-order tensor with respect to itself yielding the fourth-order identity tensor:

\[
A_{ij} = \frac{\partial A_{ij}}{\partial A_{ij}} g_i \otimes g_j \otimes g_i \otimes g_j = (AC) \otimes (DB).
\]

The properties of the fourth-order identity tensor \( I \) following from (2.25), (3.1) and (3.5) can now be given by

\[
I : B = B : I = B, I : D = D : I = D.
\]

Using (2.26), (3.7) and (3.8) the validity of the following important identities with the trace of a fourth-order tensor can also be proved:

\[
\begin{align*}
\text{tr}[C : D] & = \text{tr}[D : C] = [C : D] : I = C : D = D : C = D^T \otimes C, \\
\text{tr}[A \otimes B] & = [A \otimes B] : I = \text{tr} A \text{tr} B, \quad \text{tr}[A \times B] = [A \times B] : I = A : B, \\
\text{tr} I & = I : I = (1 : 1)(1 : 1) = 9, \quad \text{tr}(I \times I) = I \times I = I : I = 3.
\end{align*}
\]

Due to the important property

\[
(I \times I) : A = A : (I \times I) = (\text{tr} A) I
\]

the tensor \( I \times I \) introduced in (3.17) is referred to as the trace projection tensor.

**Remark.** For the standard definition of the tensor derivative the trace projection tensor has the form

\[
(I \times I)^R = (g_i \otimes g_j \otimes g_j \otimes g_i)^R = g_i \otimes g_j \otimes g_i \otimes g_j = I \otimes I.
\]
Now, we construct the partial derivative of a second-order tensor with respect to its transverse counterpart delivering the transposition projection tensor $T$:

$$A_{ij}^T = A_{ij} = \frac{\partial A_{ij}'}{\partial A_{ij}'} g' \otimes g' \otimes g_i \otimes g_j = \delta_{ij} \delta_{ij} g' \otimes g' \otimes g_i = I = T,$$  \hspace{1cm} (3.19)

which possesses the following properties:

$$T : B = g' \otimes g' \otimes g_i \otimes g_j : B_{ij} g_i \otimes g_j = B_{ij} g' \otimes g_j = B^T, \quad B : T = B^T.$$ \hspace{1cm} (3.20)

For the derivative with respect to the symmetric part $\text{sym}A = (1/2)(A + A^T)$ of a tensor $A$ one receives under consideration of (2.12) and (2.33)

$$A_{\text{sym}} = A^T_{\text{sym}} = \text{sym} A_{\text{sym}} = \frac{1}{2} (I + T) = I'.$$ \hspace{1cm} (3.21)

Generalizing this result to the derivative of an arbitrary tensor delivers:

$$B_{\text{sym}} = B_{\text{sym}} : A_{\text{sym}} = B_{\text{sym}} : I' = (B'_{A})'.$$ \hspace{1cm} (3.22)

Now, attention is focused on the tensor differentiation rules. We start with the product derivative

$$(AB)_C = A_{c} B + AB_{c},$$ \hspace{1cm} (3.23)

which is well known for scalar values. According to the new definition of the derivative (2.12) (2.31) the product derivative rule (3.23) also holds for second-order tensors.

**Proof.**

$$(AB)_c = \frac{\partial (A_{ij} B_{ij} g_i \otimes g_j)}{\partial (C_{ij} g_i \otimes g_j)} = \frac{\partial (A_{ij} B_{ij}')}{\partial (C_{ij} g_i \otimes g_j)} g_i \otimes g' \otimes g_j \otimes g_j$$

$$= \frac{\partial A_{ij}}{\partial C_{ij}} B_{ij} g_i \otimes g' \otimes g_j \otimes g_j + A_{ij} \frac{\partial B_{ij}'}{\partial C_{ij}} g_i \otimes g' \otimes g_j \otimes g_j = A_{c} B + AB_{c}.$$  

Using this remarkable relation one can easily obtain the derivative of the tensor power function with respect to its argument $B_{n}^n$ $(n = 1, 2, \ldots)$. Considering (3.3), (3.5), and (3.15) one gets:

$$B_{n}^2 = B \otimes I + I \otimes B,$$ \hspace{1cm} (3.24)

$$B_{n}^3 = B^2 \otimes I + B \otimes B + I \otimes B^2 = (B \otimes I)^2 + (B \otimes I) + (I \otimes B)^2,$$ \hspace{1cm} (3.25)

$$B_{n}^n = \sum_{r=0}^{n-1} B^{n-1-r} \otimes B^r = \sum_{r=0}^{n-1} (B \otimes I)^{n-1-r} \otimes (I \otimes B)^r,$$ \hspace{1cm} (3.26)

$$B_{n}^n : I = n B^{n-1},$$ \hspace{1cm} (3.27)

where we adopt $B^0 = I$. To obtain the derivative of the inverse function $(B^{-1})_{n}^n$ we first use the identity: $(BB^{-1})_{B} = I_{B} = 0$, which leads by virtue of (3.3) and (3.23) to

$$(B^{-1})_{n}^n = -B^{-1} \otimes B^{-1}.$$ \hspace{1cm} (3.28)

By analogue and exploiting (3.28) we can further write for the derivative of the power function with a negative integer exponent:
\[(B^n)_B = \sum_{r=0}^{n-1} B^{-r-1} \otimes B^{n-r}, \quad (B^{-n})_B : I = -nB^{-n-1}.\] (3.29)

Two other important differentiation rules can be formulated as follows:

\[ (\varepsilon A)_B = A \times \varepsilon_B + \varepsilon A_B, \quad (A : B)_C = A : B_C + B : A_C, \] (3.31)

where \(\varepsilon\) denotes a scalar function.

**Proof.**\[
\frac{\partial (\varepsilon A^i_j g^k \otimes g^l)}{\partial B^i_j} g^k \otimes g^l = \frac{\partial (\varepsilon A^i_j)}{\partial B^i_j} g^k \otimes g^l + \varepsilon \frac{\partial A^i_j}{\partial B^i_j} g^k \otimes g^l = \varepsilon A \times \varepsilon_B + \varepsilon A_B.\]

\[
(A : B)_C = \frac{\partial (A^i_j B^k_l)}{\partial (C^i_j g^k \otimes g^l)} = \left( A^i_j \frac{\partial B^k_l}{\partial C^i_j} + B^k_l \frac{\partial A^i_j}{\partial C^i_j} \right) g^k \otimes g^l = A : B_C + B : A_C.
\]

For the derivative of a scalar \(f(A)\) and a tensor-valued \(A(C)\) tensor function one can prove the following chain rules:

\[
f(A)_B = \frac{\partial f}{\partial A^i_j} \frac{\partial A^i_j}{\partial B^i_j} g^k \otimes g^l = f_{,A} : A_B,\]

\[
A_B = \frac{\partial A^i_j}{\partial C^i_j} \frac{\partial C^i_j}{\partial B^i_j} g^k \otimes g^l = A_C : C_B.
\] (3.33)

### 4. Spectral decomposition, eigenvalues and principal invariants of a fourth-order tensor

The double contraction 4:2 (2.15) establishes the basis for the formulation of the eigenvalue problems

\[D : M = \lambda M\] (4.1)

for a fourth-order tensor and enables then to determine its nine eigenvalues and principal invariants. To this end, we transform (4.1) as follows

\[(D - \lambda I) : M = 0.\] (4.2)

This equation has a non-trivial solution \(M \neq 0\) if the corresponding characteristic equation

\[
\det[D - \lambda I] = \begin{vmatrix}
D^{11}_{11} - \lambda & D^{12}_{11} & \cdots & D^{13}_{11} \\
D^{11}_{12} & D^{12}_{12} - \lambda & \cdots & D^{13}_{12} \\
\vdots & \vdots & \ddots & \vdots \\
D^{11}_{13} & D^{12}_{13} & \cdots & D^{13}_{13} - \lambda
\end{vmatrix} = 0,
\] (4.3)

which can also be presented in the form:

\[-\lambda^9 + I_D \lambda^8 - II_D \lambda^7 + III_D \lambda^6 + \cdots + IX_D = 0,\] (4.4)

is fulfilled. In this equation the coefficients \(I_D, II_D, III_D, \ldots, IX_D\) denote the principal invariants of the tensor \(D\). On the basis of Newton formula and in view of the identity \(A : B = AB\) they can be expressed in terms of principal traces of \(D\) (see also [42]):
The closed formula solution for the eigenvalue bases (4.8) takes in this case the form:

\[
\mathbf{M}_r \times \mathbf{M}_r = \prod_{s \neq r}^{6} \frac{\mathbf{D} - \lambda_r \mathbf{I}}{\lambda_r - \lambda_s} \text{ with } \prod_{i}^{n} \mathbf{A}_i = \mathbf{A}_1 : \mathbf{A}_2 : \cdots : \mathbf{A}_n.
\]  

(4.7)
Concluding this section we present for example spectral decompositions of some important fourth-order tensors $\mathbf{I}$, $\mathbf{I} \times \mathbf{I}$, $\mathbf{T}$ and $\mathbf{T}'$:

$$
I = \sum_{i,j=1}^{3} (\mathbf{n}_i \otimes \mathbf{n}_j) \times (\mathbf{n}_i \otimes \mathbf{n}_j), \quad \lambda_1 = \lambda_2 = \cdots = \lambda_9 = 1,
$$

$$
I \times I = 3 \left( \frac{1}{\sqrt{3}} \mathbf{I} \right) \times \left( \frac{1}{\sqrt{3}} \mathbf{I} \right), \quad \lambda_1 = 3, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_9 = 0,
$$

$$
T = \sum_{i}^{3} (\mathbf{n}_i \otimes \mathbf{n}_i) + \sum_{i,j=1}^{3} \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j + \mathbf{n}_j \otimes \mathbf{n}_i) \times \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j + \mathbf{n}_j \otimes \mathbf{n}_i)
$$

$$
- \sum_{i,j=1}^{3} \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j - \mathbf{n}_j \otimes \mathbf{n}_i) \times \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j - \mathbf{n}_j \otimes \mathbf{n}_i),
$$

$$
\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 1, \quad \lambda_7 = \lambda_8 = \lambda_9 = -1,
$$

$$
T' = \frac{1}{2} (I + T) = \sum_{i}^{3} (\mathbf{n}_i \otimes \mathbf{n}_i) + \sum_{i,j=1}^{3} \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j + \mathbf{n}_j \otimes \mathbf{n}_i) \times \frac{1}{\sqrt{2}} (\mathbf{n}_i \otimes \mathbf{n}_j + \mathbf{n}_j \otimes \mathbf{n}_i),
$$

$$
\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 1, \quad \lambda_7 = \lambda_8 = \lambda_9 = 0
$$

with $\mathbf{n}_i$ ($i = 1, 2, 3$) being a set of orthogonal unit vectors: $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}$.

5. Application to isotropic tensor functions

A tensor-valued tensor function $\mathbf{G}(\mathbf{A})$ is said to be isotropic if it satisfies the relation [39]:

$$
\mathbf{G}(\mathbf{Q} \mathbf{A} \mathbf{Q}^T) = \mathbf{Q} \mathbf{G}(\mathbf{A}) \mathbf{Q}^T
$$

for an arbitrary orthogonal tensor $\mathbf{Q} = \mathbf{Q}^{-T}$.

In the further derivation we shall first deal with tensor power series building a special class of isotropic tensor functions:

$$
\mathbf{G}(\mathbf{A}) = g_0 \mathbf{I} + g_1 (\mathbf{A} - z \mathbf{I}) + g_2 (\mathbf{A} - z \mathbf{I})^2 + g_3 (\mathbf{A} - z \mathbf{I})^3 + \cdots,
$$

where $g_i$ and $z$ are scalar-valued constants. These functions are referred here to as **analytical** tensor functions. For instance, isotropic tensor functions such as exponential and logarithmic ones being important and often used in computational mechanics are defined in the form (5.2) by

$$
\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!} \mathbf{A}^2 + \cdots,
$$

$$
\ln(\mathbf{A}) = (\mathbf{A} - \mathbf{I}) - \frac{1}{2} (\mathbf{A} - \mathbf{I})^2 + \frac{1}{3} (\mathbf{A} - \mathbf{I})^3 + \cdots
$$

Our purpose is to obtain the derivative of an isotropic function with respect to its argument, which is of high importance in computational elasticity and elasto-plasticity for the construction of tangent moduli. Using the relation (3.26) and setting in (5.2) without loss of generality $z = 0$, we receive for the derivative of an analytical tensor function

$$
\mathbf{G}(\mathbf{A})_{|\mathbf{A}} = g_1 \mathbf{I} + g_2 (\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}) + \cdots + g_n \sum_{r=0}^{r=n-1} \mathbf{A}^{n-1-r} \otimes \mathbf{A}^r + \cdots
$$
On the basis of expression (5.5) the important identity for the time derivative can be confirmed as follows

$$G(A)_{\dot{A}} = g_1 \dot{A} + g_2 (A \dot{A} + \dot{A} A) + \cdots + g_n \sum_{r=0}^{n-1} A^{n-r} \dot{A} A^r + \cdots = \dot{G}(A).$$

(5.6)

The representation (5.5) holds for arbitrary tensors. In the special case of a symmetric tensor argument $A = A^T$ the derivative $G(A)_{\dot{A}}$ takes the form:

$$G(A)_{\dot{A}} = g_1 I' + g_2 (A \otimes I + I \otimes A)' + \cdots + g_n \sum_{r=0}^{n-1} (A^{n-r} \otimes A')' + \cdots$$

(5.7)

and can be expressed by means of the spectral decomposition of $A$

$$A = \lambda_1 n_1 \otimes n_1 + \lambda_2 n_2 \otimes n_2 + \lambda_3 n_3 \otimes n_3,$$

(5.8)

by

$$G(A)_{\dot{A}} = \sum_{n=1}^{\infty} g_n \left[ n \sum_{a}^{3} \lambda_a^{n-1} (n_a \otimes n_a \otimes n_a) + \sum_{a,b \neq a}^{3} \lambda_a^{n-1} \lambda_b \lambda_a \lambda_b (n_a \otimes n_a \otimes n_b \otimes n_b) \right].$$

(5.9)

It can easily be seen that the derivative (5.9) presents a super-symmetric fourth-order tensor in the spirit of definition (2.11):

$$[G(A)_{\dot{A}}]^T = G(A)_{\dot{A}}, \quad [G(A)_{\dot{A}}]^i = G(A)_{\dot{A}}.$$  

(5.10)

Introducing the abbreviation for the diagonal function [24] $G(\lambda) = \sum_{n=0}^{\infty} g_n \lambda^n$ allowing the alternative representation for $G(A)$

$$G(A) = G(\lambda_1) n_1 \otimes n_1 + G(\lambda_2) n_2 \otimes n_2 + G(\lambda_3) n_3 \otimes n_3,$$

(5.11)

the relation (5.9) can be given in the case of distinct eigenvalues $\lambda_i$ in terms of eigenvectors $n_a$ by

$$G(A)_{\dot{A}} = \sum_{a}^{3} G(\lambda_a) n_a \otimes n_a \otimes n_a \otimes n_a + \sum_{a,b \neq a}^{3} \frac{G(\lambda_a) - G(\lambda_b)}{\lambda_a - \lambda_b} (n_a \otimes n_a \otimes n_b \otimes n_b)'$$

$$= \sum_{a}^{3} G(\lambda_a) n_a \otimes n_a \otimes n_a \otimes n_a + \frac{1}{2} \sum_{a,b \neq a}^{3} \frac{G(\lambda_a) - G(\lambda_b)}{\lambda_a - \lambda_b} n_a \otimes (n_a \otimes n_b + n_b \otimes n_a) \otimes n_b.$$  

(5.12)

Applying the operation $(\cdots)^R$ (2.13) to the relation (5.12) we can return $G(A)_{\dot{A}}$ to the standard definition of the tensor derivative:

$$[G(A)_{\dot{A}}]^R = \sum_{a}^{3} G(\lambda_a) n_a \otimes n_a \otimes n_a \otimes n_a + \frac{1}{2} \sum_{a,b \neq a}^{3} \frac{G(\lambda_a) - G(\lambda_b)}{\lambda_a - \lambda_b} n_a \otimes (n_a \otimes n_b + n_b \otimes n_a),$$

(5.13)

which corresponds to the well-known result [9, 27].

In terms of the eigenvalue bases $M_i = n_i \otimes n_i \otimes n_i \otimes n_i, (i = 1, 2, 3)$, which can be obtained by means of the closed-formula solution [22, 26, 36], the derivative (5.13) is expressible in the form:

$$G(A)_{\dot{A}} = \sum_{a}^{3} G(\lambda_a) M_a \otimes M_a + \sum_{a,b \neq a}^{3} \frac{G(\lambda_a) - G(\lambda_b)}{\lambda_a - \lambda_b} (M_a \otimes M_b)'.$$  

(5.14)

The last representation has the advantage, that it avoids a numerically expensive determination of eigenvectors. Comparing with the analogous solution by Miehe [22, 24] it is observable that the representation (5.14) does not include the derivatives of the eigenvalue bases and has a simpler and more compact form.

The special cases of two or three equal eigenvalues $\lambda_i$ require a special treatment. Considering in (5.14) the limit case $\lambda_a - \lambda_b = \Delta \rightarrow 0$, one can obtain the following results:
1. **Triple coalescence of eigenvalues**: \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda, \ A = \lambda I \),

\[
G(A)_{\lambda} = G'(\lambda) \lambda' .
\] (5.15)

2. **Double coalescence of eigenvalues**: \( \lambda_2 = \lambda_3 = \lambda, \ A = \lambda_a M_a + \lambda( I - M_a) \),

\[
G(A)_{\lambda_a} = G'(\lambda_a) M_a \otimes M_a + \frac{G(\lambda_a) - G(\lambda)}{\lambda_a - \lambda} \left[ M_a \otimes (I - M_a) + (I - M_a) \otimes M_a \right]' + G'(\lambda)(I - M_a) \otimes (I - M_a)' .
\] (5.16)

According to the representation theorem [39], an arbitrary isotropic tensor function, which cannot generally be presented via power series (5.2), is given in the form:

\[
G(A) = \varphi_0(A) I + \varphi_1(A) A + \varphi_2(A) A^2 ,
\] (5.17)

where \( \varphi_i(A) = \varphi_i(QAQ^T) \) are scalar-valued isotropic tensor functions of \( A \). By virtue of (3.13), (3.24) and (3.31) we construct the derivative \( G(A)_{\lambda} \) by

\[
G(A)_{\lambda} = \varphi_1(A) I + \varphi_2(A) (A \otimes I + I \otimes A) + I \times \varphi_0(A)_{\lambda} A + \varphi_1(A)_{\lambda} A + A^2 \times \varphi_2(A)_{\lambda} ,
\] (5.18)

where exploiting the isotropy of \( \varphi_i : \varphi_i(A) = \varphi_i(I_A, I_A, I_A) \) their derivatives can be given by

\[
\varphi_i(A)_{\lambda} = \left( \frac{\partial \varphi_i}{\partial I_A} + \frac{\partial \varphi_i}{\partial I_A^T} I_A \right) I - \frac{\partial \varphi_i}{\partial I_A} A^T + \frac{\partial \varphi_i}{\partial I_A} I A A^T .
\] (5.19)

Using the result (5.18) and (5.19) and by virtue of (3.1) and (3.2) it can easily be proved that the relation:

\[
(AG(A)_{\lambda}) \cdot I A = (G(A)_{\lambda} A) \cdot I A = A G(A)_{\lambda} : I \otimes A : I = I \otimes A : G(A)_{\lambda} : A \otimes I : I ,
\] (5.20)

where \( I \) presents an arbitrary second-order tensor, holds for symmetric tensors \( A = A^T \).

For the time derivative of an arbitrary isotropic tensor function we obtain by using (3.1), (3.2), (3.14), (5.18) and (5.19) the well-known relation:

\[
G(A)_{\lambda} : \dot{A} = \varphi_1(A) \dot{A} + \varphi_2(A)(A \dot{A} + \dot{A} A) + I \dot{\varphi}_0(A) + A \dot{\varphi}_1(A) + A^2 \dot{\varphi}_2(A) = \dot{G}(A) .
\] (5.21)

To complete this section we give the derivatives of some isotropic tensor functions, which cannot be expanded in power series (5.2):

\[
[\text{tr}(A)]_{\lambda} = I \times (\text{tr}A)_{\lambda} = I \times I ,
\] (5.22)

\[
(\text{dev}A)_{\lambda} = \left[ A - \frac{1}{3} (\text{tr}A) I \right]_{\lambda} = I - \frac{1}{3} I \times I = P
\] (5.23)

with \( P \) being the deviatoric projection tensor: \( A : P = P : A = \text{dev}A \).

6. **Tangent moduli**

One of the most important applications fourth-order tensors find in computational elasticity and elastoplasticity, where they appear as tangent or elastic moduli. The algebra operations and differentiation rules presented above now enable to formulate tangent moduli in absolute notation and discuss their properties. The formulation succeeds in spatial as well as material descriptions. Important notations to be exploited in this context are:
\[ \psi = \psi(C) \]  
energy density function (per unit volume of the undeformed body),

\[ \mathbf{F} = \mathbf{g} \otimes \mathbf{G} \]  
deformation gradient,

\[ \mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} \]  
right Cauchy-Green tensor,

\[ \mathbf{b} = \mathbf{FF}^\mathsf{T} \]  
left Cauchy-Green tensor (Finger tensor),

\[ \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \]  
Green–Lagrange strain tensor,

\[ \mathbf{S} \]  
second Piola–Kirchhoff stress tensor,

\[ \tau = \mathbf{FSF}^\mathsf{T} \]  
Kirchhoff stress tensor,

\[ \mathbf{C}, \mathbf{e} \]  
tangent (elastic) moduli in material and spatial description respectively,

\[ \mathbf{l} = \mathbf{FF}^{-1} \]  
spatial gradient of velocity,

\[ \mathbf{d} = (1/2)(1 + \mathbf{I}^\mathsf{T}) \]  
rate of deformation tensor,

\[ L_a \mathbf{a} = \mathbf{a} - \mathbf{la} - \mathbf{al}^\mathsf{T} \]  
Lie derivative (Oldroyd rate) of a spatial tensor \( \mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \).

We start with the general definitions of the tangent moduli. Using the Doyle–Ericksen formula [13] and by virtue of relations (3.1) and (3.12), the tangent moduli are constructed such that they satisfy the corresponding constitutive rate equations.

**Material description:**

\[
\mathbf{S} = 2 \frac{\partial \psi}{\partial \mathbf{C}} = \frac{\partial \psi}{\partial \mathbf{E}}, \tag{6.1}
\]

\[
\mathbf{C} = 4 \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} = \frac{\partial^2 \psi}{\partial \mathbf{E} \partial \mathbf{E}}, \tag{6.2}
\]

\[
\dot{\mathbf{S}} = 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} : \frac{1}{2} \dot{\mathbf{C}} = \mathbf{C} : \frac{1}{2} \dot{\mathbf{C}} = \mathbf{C} : \dot{\mathbf{E}}. \tag{6.3}
\]

**Spatial description:**

\[
\tau = 2 \mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} \mathbf{F}^\mathsf{T}, \tag{6.4}
\]

\[
\mathbf{c} = (\mathbf{F} \otimes \mathbf{F}^\mathsf{T}) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^\mathsf{T} \otimes \mathbf{F}) \quad (\mathbf{c}^{\alpha \beta \gamma \delta} = F_\alpha^\beta F_\gamma^\delta F_\epsilon^\epsilon F_\delta^\gamma \mathbf{C}^{\alpha \beta \gamma \delta}), \tag{6.5}
\]

\[
L_a \tau = \mathbf{FSF}^\mathsf{T} = \mathbf{F} \left(4 \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : \mathbf{E}\right) \mathbf{F}^\mathsf{T} = 4 \mathbf{F} \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} \mathbf{F}^\mathsf{T} : (\mathbf{F}^\mathsf{T} \mathbf{dF}) = \mathbf{c} : \mathbf{d}. \tag{6.6}
\]

The expression of the tangent modulus in the spatial formulation (6.5) coincides with the well-known representation [33]:

\[
\mathbf{c} = 2 \frac{\partial \tau}{\partial \mathbf{g}} = 4 \frac{\partial^2 \psi}{\partial \mathbf{g} \partial \mathbf{g}}, \tag{6.7}
\]

if only the derivation in (6.7) is carried out with respect to the covariant components of the metric tensor \( \mathbf{g} = g_{ij}\mathbf{g}_i \otimes \mathbf{g}_j \) in the current configuration and it is assumed that the covariant vector bases of \( \tau = \tau^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \) are independent of \( g_{ij} \). This leads to

\[
\mathbf{c} = 2 \frac{\partial \tau}{\partial \mathbf{g}} = \frac{\partial \tau^{ij}}{\partial g_{ij}} \mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}_j \otimes \mathbf{g}_j = \frac{\partial S}{\partial \mathbf{C}} \mathbf{F}^\mathsf{T} : (\mathbf{F} \otimes \mathbf{F}) = (\mathbf{F} \otimes \mathbf{F}^\mathsf{T}) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^\mathsf{T} \otimes \mathbf{F}), \tag{6.8}
\]

which is evidently identical with (6.5). Such a derivative of a tensor, where its basis is kept constant, is analogical to the Lee derivative. Otherwise, the differentiation with respect to the metric tensor in the spirit of the definition (2.31) would lead to a meaningless result, since \( \mathbf{g} \) presents the identity tensor.

Material and spatial formulations presented are valid both for isotropic and anisotropic materials. In literature there exists also a spatial representation of the elastic modulus in term of the left Cauchy–Green tensor \( \mathbf{b} \) proposed by Miehe [23] for isotropic materials

\[
\mathbf{c} = 4 \mathbf{b} \frac{\partial^2 \psi}{\partial \mathbf{b} \partial \mathbf{b}} \mathbf{b}, \quad L_a \tau = \mathbf{c} : \mathbf{d}. \tag{6.9}
\]
In the following we show, that the definition of the material tensor (6.9) leads to the rate relation (6.9) if only one assumes that the tensor \( b \) is not symmetric while derivating with respect to \( b \). To prove this we first transform the elastic modulus (6.9) to the proposed definition (2.31) of the tensor derivative. Then

\[
\mathbf{c} = 4(\mathbf{b} \otimes \mathbf{I}) : \frac{\partial^2 \psi}{\partial \mathbf{b} \mathbf{c} \partial \mathbf{b}} : (\mathbf{I} \otimes \mathbf{b}) = 4(\mathbf{I} \otimes \mathbf{b}) : \frac{\partial^2 \psi}{\partial \mathbf{b} \mathbf{c} \partial \mathbf{b}} : (\mathbf{b} \otimes \mathbf{I}),
\]

such that

\[
\mathbf{c}^R = 4 \left( b_{\alpha \beta \gamma \delta} \frac{\partial^2 \psi}{\partial b_{\alpha \beta} \partial b_{\gamma \delta}} \mathbf{g}_{\alpha \beta} \otimes \mathbf{g}_{\gamma \delta} \otimes \mathbf{g}_{r} \right)^R = 4 b_{\alpha \beta \gamma \delta} \frac{\partial^2 \psi}{\partial b_{\alpha \beta} \partial b_{\gamma \delta}} \mathbf{g}_{\alpha \beta} \otimes \mathbf{g}_{\gamma \delta} \otimes \mathbf{g}_{r} = 4 b_{\alpha \beta \gamma \delta} \frac{\partial^2 \psi}{\partial \mathbf{b} \mathbf{c} \partial \mathbf{b}}.
\]

Using the constitutive relation for isotropic materials in the form \( \tau = 2\psi, b = 2b\psi, b \) and by virtue of (2.33), (5.20) and (3.11) one can obtain:

\[
L_r \tau = \tau, b : \mathbf{b} - \mathbf{l} \tau - \tau \mathbf{T} = \left( [2 \mathbf{I} \otimes \psi, \mathbf{b}) + 2b \psi, \mathbf{b} \right) : (\mathbf{b} \mathbf{l}) + \left( [2 \psi, \mathbf{b} \otimes \mathbf{I}) + 2 \psi, \mathbf{b} \mathbf{b} \right) : (\mathbf{b} \mathbf{l} \mathbf{T}) - \mathbf{l} \tau - \tau \mathbf{T}
\]

\[
= 4(\mathbf{I} \otimes \mathbf{b}) : \psi, \mathbf{b} : (\mathbf{b} \otimes \mathbf{I}) : \mathbf{d} + 2b^T \psi, \mathbf{b} \mathbf{b} + 2 \psi, \mathbf{b} \mathbf{b} - (\mathbf{l} \tau + \tau \mathbf{T}) = \mathbf{c} : \mathbf{d} + 2b^T \psi, \mathbf{b} \mathbf{b} + 2 \psi, \mathbf{b} \mathbf{b} - \frac{1}{2} (\mathbf{l} \tau + \tau \mathbf{T}).
\]

On the contrary, assuming \( b \) is not symmetric in the differentiation and using the corresponding definition of the tensor derivative (2.31) we receive:

\[
L_r \tau = \tau, b : \mathbf{b} - \mathbf{l} \tau - \tau \mathbf{T} = \left( [2 \mathbf{I} \otimes \psi, \mathbf{b}) + 2b \psi, \mathbf{b} \right) : (\mathbf{b} \mathbf{l}) + \left( [2 \psi, \mathbf{b} \otimes \mathbf{I}) + 2 \psi, \mathbf{b} \mathbf{b} \right) : (\mathbf{b} \mathbf{l} \mathbf{T}) - \mathbf{l} \tau - \tau \mathbf{T}
\]

\[
= 4(\mathbf{I} \otimes \mathbf{b}) : \psi, : (\mathbf{b} \otimes \mathbf{I}) : \mathbf{d} + 2b^T \psi, \mathbf{b} \mathbf{b} + 2 \psi, \mathbf{b} \mathbf{b} - (\mathbf{l} \tau + \tau \mathbf{T}) = \mathbf{c} : \mathbf{d}.
\]

It can easily be proved that the tangent moduli \( \mathbf{C} \) (6.2) and \( \mathbf{e} \) (6.5) are super-symmetric according to their definitions. For \( \mathbf{C} \) it directly follows from (5.10). As to \( \mathbf{e} \) we have by using (2.9), (3.9) and (3.10)

\[
\mathbf{c}^T = \left( [\mathbf{F} \otimes \mathbf{F}^T) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^T \otimes \mathbf{F}) \right)^T = \left( [\mathbf{F} \otimes \mathbf{F}^T) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^T \otimes \mathbf{F}) \right)^T = \mathbf{c},
\]

\[
\mathbf{c}^T = \left( [\mathbf{F} \otimes \mathbf{F}^T) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^T \otimes \mathbf{F}) \right)^T = \left( [\mathbf{F} \otimes \mathbf{F}^T) : \frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}} : (\mathbf{F}^T \otimes \mathbf{F}) \right)^T = \mathbf{c}.
\]

Similarly, the symmetry can also be proved for the other elastic modulus

\[
\mathbf{A} = \frac{\partial^2 \psi}{\partial \mathbf{F} \partial \mathbf{F}}, \quad \mathbf{A} = \mathbf{A}^T,
\]

which is used in literature, too [33].

The super-symmetry of the elastic moduli \( \mathbf{C} \) (6.2) and \( \mathbf{e} \) (6.5) permits their spectral decomposition in the form (4.9). For example we get for \( \mathbf{C} \)

\[
\mathbf{C} = \sum_{r=1}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r, \quad \mathbf{M}_r = \mathbf{M}_r^T.
\]

Now, the question is what properties the elastic modulus corresponding to an isotropic material possesses and how the isotropy can be identified by tangent moduli. This problem can easily be solved for the natural state, where \( b = \mathbf{C} = \mathbf{I} \). In this case the tensor function \( \delta S(\delta \mathbf{E}) = \mathbf{C} : \delta \mathbf{E} \) should satisfy the isotropy condition:

\[
\mathbf{Q} \delta \mathbf{S} \mathbf{Q}^T = \mathbf{C} : (\mathbf{Q} \delta \mathbf{E} \mathbf{Q}^T)
\]
for an arbitrary strain variation $\delta \mathbf{E}$ and an arbitrary orthogonal tensor $\mathbf{Q} = \mathbf{Q}^{-T}$. Using the decomposition of $\delta \mathbf{E}$ in terms of $\mathbf{M}_r$:

$$\delta \mathbf{E} = \sum_{r=1}^{6} \delta \mathbf{E}_r \mathbf{M}_r \rightarrow \mathbf{Q} \delta \mathbf{E} \mathbf{Q}^T = \sum_{r=1}^{6} \delta \mathbf{E}_r (\mathbf{Q} \mathbf{M}_r \mathbf{Q}^T)$$

we obtain then from (6.14):

$$\delta \mathbf{S} = \left(\sum_{r=1}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r \right) : \left(\sum_{r=1}^{6} \delta \mathbf{E}_r \mathbf{M}_r \right) = \sum_{r=1}^{6} \lambda_r \delta \mathbf{E}_r \mathbf{M}_r. \quad (6.17)$$

Under consideration of (6.17) the condition (6.15) takes then the form:

$$\sum_{r=1}^{6} \lambda_r \delta \mathbf{E}_r \mathbf{M}_r \mathbf{Q} \mathbf{Q}^T = \sum_{r=1}^{6} \lambda_r \delta \mathbf{E}_r \left[ \mathbf{M}_r : (\mathbf{Q} \mathbf{M}_r \mathbf{Q}^T) \right] \mathbf{M}_r. \quad (6.18)$$

Contracting the left- and right-hand sides with $\mathbf{M}_r$ yields

$$\sum_{r=1}^{6} \lambda_r \delta \mathbf{E}_r \left[ \mathbf{M}_r : (\mathbf{Q} \mathbf{M}_r \mathbf{Q}^T) \right] = \sum_{r=1}^{6} \lambda_r \delta \mathbf{E}_r \left[ \mathbf{M}_r : (\mathbf{Q} \mathbf{M}_r \mathbf{Q}^T) \right] \rightarrow \sum_{r=1}^{6} (\lambda_r - \lambda_r) \delta \mathbf{E}_r \left[ \mathbf{M}_r : (\mathbf{Q} \mathbf{M}_r \mathbf{Q}^T) \right] = 0$$

$$\forall \delta \mathbf{E}_r \quad \text{and} \quad \forall \mathbf{Q} = \mathbf{Q}^{-T}. \quad (6.19)$$

Since eigentensors $\mathbf{M}_r$ are symmetric the condition (6.19) can be satisfied only in two following cases:

1. $\lambda_1 = \lambda_2 = \cdots \lambda_6 = 2\mu \rightarrow \mathbf{C} = 2\mu \mathbf{I}$, \quad (6.20)
2. $\lambda_1 = 3\lambda, \quad \mathbf{M}_1 = \frac{1}{\sqrt{3}} \mathbf{I}, \quad \lambda_2 = \cdots \lambda_6 = 0 \rightarrow \mathbf{C} = \lambda \mathbf{I} \times \mathbf{I}$, \quad (6.21)

where $2\mu$ and $\lambda$ denote material constants. Generally we can state that an elastic modulus corresponds to an isotropic material in the natural configuration, if and only if $\mathbf{C}$ can be represented via linear combination of the tensors $\mathbf{I}'$ and $\mathbf{I} \times \mathbf{I}$ in the form of the St. Venant–Kirchhoff model (see also [14,15,17,19,20,29,31]):

$$\mathbf{C} = 2\mu \mathbf{I}' + \lambda \mathbf{I} \times \mathbf{I}. \quad (6.22)$$

The analogous statement for the constitutive relations has been proved by Ciarlet [10].

### 7. Examples of elasticity tensors

Using the results (6.1), (6.2), (6.4) and (6.5) of the previous section as well as tensor algebra operations and differentiation rules (3.5), (3.6), (3.9), (3.10), (3.13)–(3.15), (3.22), (3.23), (3.28), (3.31)–(3.33), (5.21) and (5.22), we derive here tangent tensors in material and spatial descriptions for a number of hyperelastic material models intensively used in continuum mechanics.

1. **St. Venant–Kirchhoff material model**:

$$\psi = \frac{1}{2} \kappa (\text{tr} \mathbf{E})^2 + \mu \text{tr} (\text{dev} \mathbf{E})^2,$$

$k$ and $\mu$ being the compression and shear moduli respectively.

**Material description**:

$$\mathbf{S} = \frac{\partial \psi}{\partial \mathbf{E}} = \kappa (\text{tr} \mathbf{E}) \mathbf{I} + \mu \frac{\partial (\text{tr} \mathbf{E}) (\text{dev} \mathbf{E})^2}{\partial (\text{dev} \mathbf{E})} : \frac{\partial (\text{dev} \mathbf{E})}{\partial \mathbf{E}} = \kappa (\text{tr} \mathbf{E}) \mathbf{I} + 2\mu \text{dev} \mathbf{E}, \quad (7.2)$$

$$\mathbf{C} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \kappa \mathbf{I} \times \mathbf{I} + 2\mu \left( \mathbf{I}' - \frac{1}{3} \mathbf{I} \times \mathbf{I} \right). \quad (7.3)$$
Spatial description:
\[
\mathbf{\tau} = \mathbf{FSF}^T = \frac{1}{2} \kappa (\text{tr} \, \mathbf{b} - 3) \mathbf{b} + \mu \mathbf{b} \text{dev} \, \mathbf{b},
\]
(7.4)

\[
\mathbf{c} = (\mathbf{F} \otimes \mathbf{F}^T) : \mathbf{C} : (\mathbf{F}^T \otimes \mathbf{F}) = \kappa \mathbf{b} \times \mathbf{b} + 2\mu \left[(\mathbf{b} \otimes \mathbf{b})' - \frac{1}{3} \mathbf{b} \otimes \mathbf{b}\right].
\]
(7.5)

2. Mooney–Rivlin material model [25]:
\[
\psi = C_1(I_C - 3) + C_2(\Pi_C - 3) = C_1(I_b - 3) + C_2(\Pi_b - 3),
\]
(7.6)

where \(C_1\) and \(C_2\) are material constants.

Material description:
\[
\mathbf{S} = 2 \frac{\partial \psi}{\partial \mathbf{C}} = 2C_1 \mathbf{I} + 2C_2(\mathbf{I}_c \mathbf{I} - \mathbf{C}),
\]
(7.7)

\[
\mathbf{C} = 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} = 4C_2(\mathbf{I} \times \mathbf{I} - \mathbf{I}')
\]
(7.8)

Spatial description:
\[
\mathbf{\tau} = \mathbf{FSF}^T = \frac{\partial \psi}{\partial \mathbf{b}} \mathbf{b} = 2C_1 \mathbf{b} + 2C_2(\mathbf{I}_c \mathbf{b} - \mathbf{b}^2),
\]
(7.9)

\[
\mathbf{c} = (\mathbf{F} \otimes \mathbf{F}^T) : \mathbf{C} : (\mathbf{F}^T \otimes \mathbf{F}) = 4C_2 \left[\mathbf{b} \times \mathbf{b} - (\mathbf{b} \otimes \mathbf{b})'\right].
\]
(7.10)

3. Ogden material model [27]:
\[
\psi(\lambda_i) = \sum_r \frac{H_r}{\varpi_r} \left(\lambda_i^{2r} + \lambda_2^{2r} + \lambda_3^{2r} - 3\right),
\]
(7.11)

where \(\lambda_i\) denote the eigenvalues of the right stretch tensor \(\mathbf{U}\) and \(\varpi_r, \mu_r\) are material parameters.

Material description:
\[
\mathbf{S} = 2 \left[(a_1 + a_{II} \mathbf{I}_C) \mathbf{I} - a_{III} \mathbf{C} + a_{III} \mathbf{III}_C \mathbf{C}^{-1}\right],
\]
(7.12)

\[
\mathbf{C} = 4 \left[(c_{II} + 2c_{II} \mathbf{I}_C + c_{III} \mathbf{II}_C + b_{II}) \mathbf{I} \times \mathbf{I} + c_{III} \mathbf{I} \times \mathbf{C} + \mathbf{C} \times \mathbf{I} + \mathbf{III}_C(b_{III} + c_{III} \mathbf{III}_C) \mathbf{C}^{-1} \times \mathbf{C}^{-1} - (c_{III} + c_{III} \mathbf{I}_C) \mathbf{I} \times \mathbf{C} + \mathbf{C} \times \mathbf{I} + \mathbf{III}_C(b_{III} + c_{III} \mathbf{III}_C) \mathbf{C}^{-1} \times \mathbf{I}
\]

Using the alternative representation form of the energy function \(\psi = \psi(\mathbf{I}_C, \mathbf{II}_C, \mathbf{III}_C)\) the unknown coefficients appearing in (7.12)–(7.15)
\[
a_K = b_K = \frac{\partial^2 \psi}{\partial K L}, \quad c_{KL} = \frac{\partial^2 \psi}{\partial K L L}, \quad K, L = \mathbf{I}_C, \mathbf{II}_C, \mathbf{III}_C
\]
(7.16)

\[\text{can be specialized for an arbitrary isotropic material model. For the Valanis–Landel hypothesis [40] and its special case Ogden model (7.11) these coefficients have been determined by Basar and Itskov [1] for the cases of distinct as well as equal eigenvalues without explicit solving the eigenvalue problems.}\]
4. Material models with decoupled volumetric–isochoric response. For these models the energy function is assumed to be additively split into volumetric and isochoric parts:

\[ \psi = U(J) + W(\mathbf{C}), \]  

(7.17)

where

\[ J = (\text{III}_C)^{1/2}, \quad \mathbf{C} = J^{-2/3} \mathbf{C}. \]  

(7.18)

Accordingly the corresponding stress tensors and elastic moduli permit the additive decomposition:

\[ \mathbf{S} = \mathbf{S}_{\text{vol}} + \mathbf{S}_{\text{iso}}, \quad \mathbf{C} = \mathbf{C}_{\text{vol}} + \mathbf{C}_{\text{iso}}, \quad \mathbf{\tau} = \mathbf{\tau}_{\text{vol}} + \mathbf{\tau}_{\text{iso}}, \quad \mathbf{c} = \mathbf{c}_{\text{vol}} + \mathbf{c}_{\text{iso}} \]  

(7.19)

and take the following form:

Material description:

\[ \mathbf{S}_{\text{vol}} = 2 \frac{\partial U}{\partial \mathbf{C}} = U'J \mathbf{C}^{-1}, \]  

(7.21)

\[ \mathbf{S}_{\text{iso}} = 2 \frac{\partial W}{\partial \mathbf{C}} - 2 \frac{\partial W}{\partial \mathbf{C}} \mathbf{C} \mathbf{S}_{\text{iso}} : \mathbf{p} = \text{dev}(\mathbf{S}_{\text{iso}} \mathbf{C}) \mathbf{C}^{-1} = \mathbf{C}^{-1} \text{dev}(\mathbf{C} \mathbf{S}_{\text{iso}}), \]  

(7.22)

\[ \mathbf{C}_{\text{vol}} = 2 \frac{\partial \mathbf{S}_{\text{vol}}}{\partial \mathbf{C}} = (U''J^2 + U'J) \mathbf{C}^{-1} \times \mathbf{C}^{-1} - 2U'J(\mathbf{C}^{-1} \times \mathbf{C}^{-1})', \]  

(7.23)

\[ \mathbf{C}_{\text{iso}} = 2 \frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} = 2\mathbf{C}^{-1} \mathbf{p}' : \left[ \mathbf{C} \frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} + (\mathbf{I} \otimes \mathbf{S}_{\text{iso}}) \right] : \mathbf{p} - 2(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1})' \text{dev}(\mathbf{C} \mathbf{S}_{\text{iso}}) \]  

\[ = \mathbf{p}' : \left[ \mathbf{C}_{\text{iso}} + 2(\mathbf{C}^{-1} \otimes \mathbf{S}_{\text{iso}})' \right] : \mathbf{p} - 2(\mathbf{C}^{-1} \otimes \mathbf{S}_{\text{iso}})', \]  

(7.24)

where

\[ \mathbf{S}_{\text{iso}} = 2 \frac{\partial W}{\partial \mathbf{C}}, \quad \mathbf{C}_{\text{iso}} = 2 \frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}} = 4 \frac{\partial^2 W}{\partial \mathbf{C}^T}, \]  

\[ \mathbf{p} = \frac{\partial \mathbf{C}}{\partial \mathbf{C}} = \mathbf{C} \times \frac{\partial J^{-2/3}}{\partial \mathbf{C}} + J^{-2/3} \mathbf{I}' = -\frac{1}{3} \mathbf{C} \times \mathbf{C}^{-1} + J^{-2/3} \mathbf{I}'. \]  

Spatial description:

\[ \mathbf{\tau}_{\text{vol}} = \mathbf{F} \mathbf{S}_{\text{vol}} \mathbf{F}^T = U'J \mathbf{I}, \]  

(7.26)

\[ \mathbf{\tau}_{\text{iso}} = \mathbf{F} \mathbf{S}_{\text{iso}} \mathbf{F}^T = \text{dev}\mathbf{\tau}_{\text{iso}}, \]  

(7.27)

\[ \mathbf{c}_{\text{vol}} = (\mathbf{F} \otimes \mathbf{F}^T) : \mathbf{C}_{\text{vol}} : (\mathbf{F}^T \otimes \mathbf{F}) = (U''J^2 + U'J) \mathbf{I} \times \mathbf{I} - 2U'J \mathbf{I}', \]  

(7.28)

\[ \mathbf{c}_{\text{iso}} = (\mathbf{F} \otimes \mathbf{F}^T) : \mathbf{C}_{\text{iso}} : (\mathbf{F}^T \otimes \mathbf{F}) = \mathbf{p}' : \left[ \mathbf{C}_{\text{iso}} + 2(\mathbf{I} \otimes \mathbf{\tau}_{\text{iso}})' \right] : \mathbf{p} - 2(\mathbf{I} \otimes \mathbf{\tau}_{\text{iso}})', \]  

(7.29)

where

\[ \mathbf{\tau}_{\text{iso}} = \mathbf{F} \mathbf{S}_{\text{iso}} \mathbf{F}^T, \quad \mathbf{c}_{\text{iso}} = (\mathbf{F} \otimes \mathbf{F}^T) : \mathbf{C}_{\text{iso}} : (\mathbf{F}^T \otimes \mathbf{F}), \quad \mathbf{F} = J^{-1/3} \mathbf{F}. \]  

(7.30)

It can easily be seen that after the transformation \((\cdots)^R\) the expressions (7.23) and (7.24) coincide with those given by Simo and Taylor [34]. In the spatial description we have forgone to use the definition (6.9)
for the isochoric moduli $\tau_{\text{iso}}$. For this reason the representation (7.26)–(7.29) holds in contrast to the analogous result by Miehe [23] not only for isotropic but also for anisotropic materials.

8. Conclusion

In the paper we have presented a theory of fourth-order tensors. The theory enables to solve many relevant problems in the treatment and application of fourth-order tensors, which are still open in the literature.

Starting with the set of notations and definitions we have presented in absolute notation the tensor algebra and differentiation rules, which are applied then to isotropic tensor functions. The important result to be emphasised is a closed-formula solution for the derivative of an isotropic tensor function, which is given in terms of eigenvalue bases for the cases of distinct and equal eigenvalues. This solution has a simpler and more compact form in contrast to those known in the literature.

The next important topic are the eigenvalue problems and the spectral decomposition of a fourth-order tensor. It is shown that the eigenvalue problem of a fourth-order tensor can be reduced to that one of its matrix representation and then easily solved. This leads to a spectral decomposition involving nine eigenvalues and nine corresponding eigentensors. In the case of super-symmetry characterizing for example some tangent moduli, the spectral decomposition can be given through only six symmetric eigentensors. Concerning the invariants of a fourth-order tensor we have come to their total number of 27, which can be obtained using various contraction rules with a second-order tensor.

A most important application of the fourth-order tensors are tangent moduli. The tensor algebra and differentiation rules presented simplify considerably the formulation and derivation of tangent moduli, which has been demonstrated on examples of some hyperelastic material models. The elastic moduli are obtained in a material as well as in a spatial description and compared with the results available in the literature.

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