This exam is closed book and notes. Show partial solutions to get partial credit.
If your answers are not written legibly, you won’t get full credit.
Clarity and succinctness will be rewarded.

Question 1: ____________(out of 15)
Question 2: ____________(out of 10)
Question 3: ____________(out of 10)
Question 4: ____________(out of 10)
Question 5: ____________(out of 15)
Question 6: ____________(out of 10)
Question 7: ____________(out of 15)
Question 8: ____________(out of 15)

Total: ____________(out of 100)
1. (a) Give three deterministic FAs over alphabet \{0, 1\} that accept the languages 
\{\}, \{\lambda\}, and \{0\}, respectively.

Note that each state must have exactly one transition labeled 0 and 1. “Trap states” might be needed!

\begin{center}
\begin{tikzpicture}
    \node[state, initial] (start) at (0,0) {start};
    \node[state] (s1) at (1,0) {0,1 start};
    \node[state] (s2) at (2,0) {0,1 start};
    \node[state] (s3) at (3,0) {0};

    \draw[->] (start) to [loop above] node {0,1} (s1);
    \draw[->] (s1) to node {0,1} (s2);
    \draw[->] (s2) to node {0,1} (s3);
    \draw[->] (s3) to node {0} (s3);
    \draw[->] (s3) to node {1} (s3);
    \draw[->] (s3) to node {0,1} (s3);
\end{tikzpicture}
\end{center}

(b) What is wrong in the following statement?:

“If each of the languages \(L_1, L_2, \ldots\) is regular, then \(\bigcup_{i=1}^{\infty} L_i\) is regular as well, because regular languages are closed under union.”

Regular sets are closed under finite union but not infinite union.

(c) Re. the above, give an example where the \(L_i\) are all regular and \(\bigcup_{i=1}^{\infty} L_i\) is not.

\(L_i = \{a^i b^i\}\) which is finite and thus regular.
But \(\bigcup_{i=1}^{\infty} L_i = \{a^i b^i : i \geq 1\}\) is not regular.

(d) What “construction” was used to show that regular languages are closed under intersection.

Cross product construction
2. What language does each of the following grammars generate? In each case, give a precise mathematical expression for the language generated by the grammar. (Hint: Find some example words that are generated. What is the shortest word generated?)

(a) 

\[ S \rightarrow Sb \mid aSb \mid \lambda \]

\[ \{a^ib^j : i \leq j\} \]

(b) 

\[ S \rightarrow Sb \mid aSb \mid b \]

\[ \{a^ib^j : i < j\} \]
3. Convert the following FA into a regular expression. Use the algorithm given in class that eliminates one node at a time and creates arcs that are labeled with regular expressions. First eliminate state 2 and then state 3.

\[
(ab + (b + aa)a^*b) (b + ab + a^{\geq 2}b)^*
= (ab + (b + aa)a^*b)(a^*b)^*
= (a^+b + ba^*b)(a^*b)^*
= (a + b)(a^*b)^+
\]
4. Use the “subset construction method” to convert the following NFA into an FA. In the resulting FA, label the states by the appropriate subset of states of the initial machine. Make sure to indicate the start state and the final states in the resulting FA.

![Diagram of the NFA](image)

![Diagram of the resulting FA](image)
5. Minimize the following FA. Show your work (give the table).
Show the resulting FA (if the number of states was reduced).
6. You are given two arbitrary $\lambda$NFAs $M_1$ and $M_2$ accepting the languages $L_1$ and $L_2$, respectively.

You are supposed to construct some new $\lambda$NFA’s from $M_1$ and $M_2$. In each case give a picture of your construction and a short description of what you did to the original machines to arrive at the new one.

(a) Construct a $\lambda$NFA from $M_1$ and $M_2$ that accepts the language $L_1 \cup L_2$?

Create new start state, connect it with $\Lambda$-transitions with both old start states. Accepting states of the new machine are the union of accepting states from $M_1$ and $M_2$.

(b) Construct a $\lambda$NFA from $M_1$ and $M_2$ that accepts the language $L_1 \cdot L_2$?

$\Lambda$-transitions from accepting states of $M_1$ to the start state of $M_2$. The new start state is the start state of $M_1$ the new accepting states are the accepting states of $M_2$.

(c) Construct a $\lambda$NFA from $M_1$ that accepts the language $(L_1)^*$?

Create new start state $q_k$, that will also be the only accepting state. $\Lambda$-transitions from $q_k$ to old start state, and from old accepting states back to $q_k$. 


7. Show that the language $L = \{a^n ba^{2n} : n \geq 0\}$ is non-regular by either

(a) exhibiting an infinite set of pairwise distinguishable words
   (you need to show that any pair in the infinite set is distinguishable)

(b) or by using the Pumping Lemma.

As a reminder, the Pumping Lemma says:

- For every regular language $L$ there is a constant $N$ such that each word $x \in L$ of length at least $N$ can be written as $uvw$ such that the following holds:
  - $|uv| \leq N$,
  - $v$ is not the empty word and
  - for all $i \geq 0$, $uv^i w \in L$.

Solution (a):

The set $\{a^n : n \geq 0\}$ is pairwise $L$-distinguishable: For any $a^i, a^j$ such that $0 \leq i < j$, take $z = ba^{2i}$. Then

$$a^i z = a^i ba^{2i} \in L$$

and

$$a^j z = a^j ba^{2i} \notin L.$$

So, having found an infinite pairwise $L$-distinguishable set, we know that $L$ is not regular.

Solution (b):

Suppose that $L$ is regular. Let $N$ be the constant from Pumping Lemma. Take a word

$$x = a^N ba^{2N}$$

and let $x = uvw$, as in pumping lemma. Since $|uv| \leq N$ and $|v| > 0$, we have $v = a^k$, where $0 < k \leq N$. By Pumping Lemma, $uv \in L$ (take $i = 0$). But

$$uw = a^{N-k} ba^{2N},$$

and $2(N-k) \neq 2N$, so $uw$ cannot be in $L$. Contradiction. Therefore, $L$ is not regular.
8. Let $L$ be a language over the alphabet $\{0,1\}$. Define $L_{00}$ as the following subset of $L$:

$$L_{00} = \{w : w \in L \text{ and } w \text{ contains two consecutive 0's}\}.$$ 

(a) Show that if $L$ is regular then $L_{00}$ is regular as well.

Assume that $L$ is regular. The language of all words containing two consecutive 0’s is described by $(0 + 1)^*00(0 + 1)^*$, so it’s regular. We can describe $L_{00}$ as

$$L_{00} = L \cap (0 + 1)^*00(0 + 1)^*.$$ 

Since intersection is a closure property of regular languages, $L_{00}$ is regular.

(b) Now suppose that $L_{00}$ is not regular.

Assuming that the implication given in part (a) holds, what do we know about $L$ in this case?

The contrapositive of part (a) says that if $L_{00}$ is not regular, then $L$ is not regular. So, in this case by contrapositive we know that $L$ is not regular.