The claim is that the following procedure cannot exist:

Procedure $H(\text{LIST}, \text{ARG})$

Specification of what $H(\text{LIST}, \text{ARG})$ is to do:

If $\text{LIST}$ is the listing of a procedure $P$, then the procedure $H$ does the following:

If procedure $P$ is run with input $\text{ARG}$ and stops
then $H$ outputs $\text{YES}$ and stops
otherwise it outputs $\text{NO}$ and stops.

Proof by contradiction: Assume a procedure $H$ fulfilling the above specification exists. Then it is easy to construct the following new procedure with the specification given:

Procedure $NH(\text{LIST})$

Specification:

If the execution of $H(\text{LIST}, \text{LIST})$ outputs $\text{YES}$
then $NH(\text{LIST})$ runs into an infinite loop.

If the execution of $H(\text{LIST}, \text{LIST})$ outputs $\text{NO}$
then $NH(\text{LIST})$ stops.

How is $NH(\text{LIST})$ constructed? Simply use the code for $H$ except that

- each occurrence of $\text{ARG}$ in the code of $H$ is replaced by $\text{LIST}$,
- each statement in the code that outputs $\text{YES}$ is replaced by a statement that causes an infinite loop,
- and each statement that outputs $\text{NO}$ is simply deleted causing the machine to stop without outputting anything.

Clearly if a procedure $H$ exists following its specification then $NH$ follows its specification.

Now consider the execution $H(<\text{NH}>,<\text{NH}>)$, where $<\text{NH}>$ is a listing of the procedure $NH$. From the specification of $H$ we know that the execution $H(<\text{NH}>,<\text{NH}>)$ either outputs $\text{YES}$ or $\text{NO}$.

Case $H(<\text{NH}>,<\text{NH}>)$ outputs $\text{YES}$: Then by the specification/construction of $NH$, the execution $NH(<\text{NH}>)$ loops. Thus by the specification of $H$, the execution $H(<\text{NH}>,<\text{NH}>)$ outputs $\text{NO}$. Contradiction!

Case $H(<\text{NH}>,<\text{NH}>)$ outputs $\text{NO}$: Then by the specification/construction of $NH$, the execution $NH(<\text{NH}>)$ stops. Thus by the specification of $H$, the execution $H(<\text{NH}>,<\text{NH}>)$ outputs $\text{YES}$. Contradiction again!

Since both case let to a contradiction we are done and the procedure $H$ with the above specification cannot exist.