1. **Solution 1.** Let $n$ be an integer satisfying $n \geq 1$, and consider the set $S_n = \{1, 2, \ldots, n\}$. To count the number of even-sized subsets of $S_n$, I will establish a one-to-one correspondence between them and the subsets of $S_{n-1} = \{1, 2, \ldots, n-1\}$. We know (thanks to the hint given in the problem) that the number of subsets of $S_{n-1}$ is equal to $2^{n-1}$; thus, establishing this one-to-one correspondence will complete the proof.

The correspondence $f : 2^{S_{n-1}} \rightarrow 2^{S_n}$ is defined as follows:

$$f(A) = \begin{cases} A & \text{if } |A| \text{ is even} \\ A \cup \{n\} & \text{if } |A| \text{ is odd}. \end{cases}$$

Clearly, the outputs of $f$ are always even-sized subsets of $S_n$.

It remains to show that $f$ is a one-to-one correspondence; in other words, I must prove that $f$ is both injective and surjective.

**Injectivity.** Let $A, B \subseteq S_{n-1}$. The proof is by contraposition, so assume that $A \neq B$; I will show that $f(A) \neq f(B)$. There are three cases:

- If $|A|$ is even and $|B|$ is even, then $f(A) = A \neq B = f(B)$.
- If $|A|$ is odd and $|B|$ is odd, then $f(A) = A \cup \{n\}$ and $f(B) = B \cup \{n\}$. Since neither $A$ nor $B$ contains $n$, we can infer from $A \neq B$ that $f(A) \neq f(B)$.

\[\textit{For purposes of surjectivity, we consider the co-domain of } f \textit{to be the set of even-sized subsets of } S_n.\]
If \(|A|\) and \(|B|\) have different parity, then \(n\) belongs to one of \(f(A)\) and \(f(B)\), but not both. Therefore, again \(f(A) \neq f(B)\).

**Surjectivity.** Let \(A \subseteq S_n\), and assume that \(|A|\) is even. There are two cases: if \(n \in A\), then we see that \(f(A - \{n\}) = A\); on the other hand, if \(n \notin A\), then we see that \(f(A) = A\).

Just to recap, we have now established that there is a one-to-one correspondence between the subsets of \(S_{n-1}\) and the even-sized subsets of \(S_n\). Therefore, since there are \(2^{n-1}\) subsets of \(S_{n-1}\), we conclude that there must also be \(2^{n-1}\) even-sized subsets of \(S_n\).

**Solution 2.** Here is another solution that, while less elementary, is also very elegant. It uses the Binomial Theorem, which states that

\[
(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},
\]

where \(\binom{n}{i}\) is equal to the number of subsets \(C \subset S_n = \{1, 2, ..., n\}\) of size \(|C| = i\). Taking \(x = y = 1\) tells us that

\[
2^n = (1 + 1)^n = \sum_{i=0}^{n} \binom{n}{i} = |2^{S_n}|.
\]

Moreover, taking \(x = -1\) and \(y = 1\), we get

\[
0 = ((-1) + 1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i = \sum_{\text{even } i \leq n} \binom{n}{i} - \sum_{\text{odd } i \leq n} \binom{n}{i}.
\]

So, the number of even subsets of \(S_n\) is equal to the number of odd subsets of \(S_n\), and therefore has to be \(\frac{1}{2}|2^{S_n}| = \frac{1}{2}2^n = 2^{n-1}\).

2. (Problem 2.1(h))
3. (Modified version of problem 2.10) The machine transitions are the same for parts (a), (b) and (c), and they are given in the diagram below.

(a) The final states are \{A1, A3, B2, B3, C3\}.

(b) The final states are \{A3, B3\}.

(c) The final states are \{A1, B2\}.

4. (Modified version of problem 2.13) The automaton is reproduced in the figure below. I have added names to the states.
• The state $q_0$ is the starting state, so for example the empty word $x_0 = \Lambda$ leads to it.
• We can get to state $q_1$ with word $x_1 = a$.
• We can get to $q_2$ with $x_2 = aa$.
• Finally, word $x_3 = ab$ leads to state $q_3$.

Now, we have to show that set $S = \{x_0, x_1, x_2, x_3\} = \{\Lambda, a, aa, ab\}$ is pairwise $L$-distinguishable ($L$ is the language accepted by the FA).

For every pair of distinct words $(x_i, x_j)$ from $S$, we will demonstrate a word $z$ such that $x_i z \in L$ and $x_j z \notin L$. There are $\binom{4}{2} = 6$ pairs to analyze. Since $x_3 = ab$ is the only word from $S$ that is accepted by the FA, we can use $z = \Lambda$ for pairs $(ab, \Lambda)$, $(ab, a)$, $(ab, aa)$ (because, $x_3 \Lambda = ab \in L$ but $x_0 \Lambda = \Lambda \notin L$, etc.). On the other hand, for pairs $(a, \Lambda)$, $(a, aa)$ we can use $z = b$. Finally, for the pair $(\Lambda, aa)$ we use $z = ab$. We demonstrated $L$-distinguishability for all 6 pairs, therefore set $S$ is pairwise $L$-distinguishable. So, by Theorem 2.21 from the textbook, the given FA has minimum number of states.

5. (Modified version of problem 2.17)

(a) The string $x = \Lambda$ is not $L$-distinguishable from the string $y = ab$.

Proof: Let $z \in \Sigma^*$ be an arbitrary string. Notice that $n_a(xz) = n_a(z)$ and $n_b(xz) = n_b(z)$ and $n_a(yz) = n_a(z) + 1$ and $n_b(yz) = n_b(z) + 1$. Therefore,

$$xz \in L$$

$$\iff n_a(xz) = n_b(xz) \quad \text{def'n of } L$$

$$\iff n_a(z) = n_b(z) \quad \text{simple substitution}$$

$$\iff n_a(z) + 1 = n_b(z) + 1 \quad \text{add 1 to both sides}$$

$$\iff n_a(yz) = n_b(yz) \quad \text{more substitution}$$

$$\iff yz \in L \quad \text{def'n of } L.$$ 

We can conclude that $x$ and $y$ are not $L$-distinguishable.

(b) Consider the set $S = \{a^n \mid n \in \mathbb{N}\}$. I will prove that the members of $S$ are pairwise $L$-distinguishable.

Consider an arbitrary pair $a^i, a^j$ where $i \neq j$. (Let us use $x$ to refer to $a^i$ and $y$ to refer to $a^j$.) Now consider the specific string
6. (Problem 2.21(h)) Consider an arbitrary pair \(a^i, a^j\) of elements in \(\{a^n \mid n \geq 0\}\) (and assume \(i \neq j\)). I will prove that the string \(x = a^i\) is \(L\)-distinguishable from the string \(y = a^j\).

Now consider the specific string \(z = ba^i\). We can see that \(xz \in L\) because \(n_a(xz) = i = n_b(xz)\); however, \(yz \notin L\) because \(n_a(yz) = j \neq i = n_b(yz)\).

7. (Extra credit.) (Problem 2.18) If \(n\) is odd, then \(L\) is empty and thus can be accepted by a machine with one state. Otherwise, if \(n\) is even, then the minimum number of states is \((k + 1)^2 + 1\), where \(n = 2k\). (Let us use \(N\) to refer to the quantity \((k + 1)^2 + 1\).) The remainder of this solution focuses on this case where \(n\) is even.

To prove that \(N\) is the minimum number, I must exhibit a set \(S\) of \(N\) strings that are pairwise \(L\)-distinguishable, because this will guarantee that every machine for \(L\) must have at least \(N\) states. Furthermore, I must show that every string is \(L\)-indistinguishable from some member of \(S\), because this will guarantee the existence of a machine with no more than \(N\) states.

The set \(S = \{a^ib^j \mid i \leq k \text{ and } j \leq k\} \cup \{a^{n+1}\}\) will work nicely.

First, we can easily check that \(|S| = N\) because \(i\) and \(j\) may each take one of \(k + 1\) values, resulting in \((k + 1)^2\) distinct strings; the extra string \(a^{n+1}\) increases this number by 1.

Next, I will show that the members of \(S\) are pairwise \(L\)-distinguishable. Before we begin, it will be helpful formulate \(L\) in a slightly different (but equivalent) way: \(L = \{x \in \{a, b\}^* \mid n_a(x) = n_b(x) = k\}\)

**Case 1.** Let \(x = a^{i_1}b^{j_1}\) and \(y = a^{i_2}b^{j_2}\), where either \(i_1 \neq i_2\) or \(j_1 \neq j_2\).

Without loss of generality, let us assume that \(i_1 \neq i_2\).
Consider the string \( z = a^{k-i_1}b^{k-j_1} \). Clearly, \( xz \in L \) because \( n_a(xz) = i_1 + k - i_1 = k = j_1 + k - j_1 = n_b(xz) \). Additionally, \( yz \notin L \) because \( n_a(yz) = i_2 + k - i_1 \neq k \).

**Case 2.** Let \( x = a^ib^j \) and \( y = a^{n+1} \), and consider the string \( z = a^{k-i}b^{k-j} \). Then \( xz \in L \) as seen in the previous case, and \( yz \notin L \) because \( |yz| > n \).

It remains to show that all strings are \( L \)-indistinguishable from some member of \( S \).

To this end, let \( x \in \{a, b\}^* \) be an arbitrary string, and let \( i = n_a(x) \) and \( j = n_b(x) \).

**Case 1.** If both \( i \leq k \) and \( j \leq k \), then \( x \) is \( L \)-indistinguishable from the string \( y = a^ib^j \).

To see this, let \( z \) be an arbitrary string. Then

\[
\begin{align*}
xz \in L \iff & \quad n_a(xz) = n_b(xz) = k \quad \text{def’n of } L \\
\iff & \quad i + n_a(z) = j + n_b(z) = k \\
\iff & \quad n_a(yz) = n_b(yz) = k \\
\iff & \quad yz \in L \quad \text{def’n of } L.
\end{align*}
\]

**Case 2.** If either \( i > k \) or \( j > k \), then \( x \) is \( L \)-indistinguishable from the string \( y = a^{n+1} \). This is obvious because for every choice of \( z \), we have \( xz \notin L \) and \( yz \notin L \).