HW1 Solutions

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January 13, 2015

1. (Problem 1.6) Let $s$ be a string of length 4 over the alphabet \{a, b\}. I will show that $s$ contains a substring $xx$ (for some nonnull string $x$).

There are two cases to consider.

- Case 1: assume that there is a double-letter (i.e. $aa$ or $bb$) in $s$ somewhere. Then $s$ contains either $aa$ or $bb$ as a substring, both of which have the required form.
- Case 2: assume that there is NOT a double-letter anywhere in $s$. In this case, the letters of $s$ must strictly alternate, so either $s = abab$ or $s = baba$. By letting $x = ab$ or $x = ba$ as appropriate, we can see that $s = xx$, and thus $s$ contains a substring of the required form.

In all cases, $s$ has a substring of the form $xx$. Since $s$ was selected arbitrarily, we conclude that all strings of length 4 contain a substring of this form.

2. (Problem 1.15)

(a) Under what circumstances are they equal? $2^{A \cup B} = 2^A \cup 2^B$ if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof: First, I will prove that “either $A \subseteq B$ or $B \subseteq A$” implies “$2^{A \cup B} = 2^A \cup 2^B$.”

Without loss of generality (due to the symmetry of the problem), we may assume that $A \subseteq B$. Note that $A \cup B = B$ in this case. Next, to show that $2^{A \cup B} = 2^A \cup 2^B$, I will show (for
every $S$) that $S \in 2^{A \cup B}$ if and only if $S \in 2^A \cup 2^B$.

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Next, I must prove that “$2^{A \cup B} = 2^A \cup 2^B$” implies “either $A \subseteq B$ or $B \subseteq A$.”

The proof is by contradiction, so assume that $2^{A \cup B} = 2^A \cup 2^B$ and $\neg(A \subseteq B)$ and $\neg(B \subseteq A)$. These last two assumptions guarantee us that there exist objects $a$ and $b$ such that $a \in A$ but $a \notin B$, and $b \in B$ but $b \notin A$. Then we can see that ${a, b} \in 2^{A \cup B}$ because $a, b \in A \cup B$, but ${a, b} \notin 2^A \cup 2^B$ because $a \notin B$ and $b \notin A$. This contradicts the assumption that $2^{A \cup B} = 2^A \cup 2^B$.

**Is one necessarily a subset of the other?** Yes, $2^A \cup 2^B$ is always a subset of $2^{A \cup B}$.

Proof: I will show that every member of $2^A \cup 2^B$ is also a member of $2^{A \cup B}$.

To this end, let $S$ be an arbitrary member of $2^A \cup 2^B$. By definition of $\cup$, this means either $S \in 2^A$ or $S \in 2^B$. Without loss of generality, we may assume that $S \in 2^A$. This means that $S \subseteq A$, which means that every member of $S$ is a member of $A$. Since every member of $S$ is also a member of $A$, we can see that every member of $S$ is also a member of $A \cup B$. Thus, $S \subseteq A \cup B$, showing us that $S \in 2^{A \cup B}$ as desired.

*(b) Under what circumstances are they equal?* They are always equal.

Proof: I will show (for every $S$) that $S \in 2^{A \cap B}$ if and only if
$S \in 2^A \cap 2^B$.

$S \in 2^{A \cap B}$

$\iff S \subseteq A \cap B$  
\text{defn of power set}

$\iff \forall x \in S, \ x \in A \cap B$  
\text{defn of } \subseteq

$\iff \forall x \in S, \ (x \in A \text{ and } x \in B)$  
\text{defn of } \cap

$\iff (\forall x \in S, \ x \in A) \text{ and } (\forall x \in S, \ x \in B)$  
\text{property of } \forall \text{ and } \land

$\iff S \subseteq A \text{ and } S \subseteq B$  
\text{defn of } \subseteq

$\iff S \in 2^A \text{ and } S \in 2^B$  
\text{defn of power set}

$\iff S \in 2^A \cap 2^B$  
\text{defn of } \cap.

\textbf{Is one necessarily a subset of the other?} Yes, they are each subsets of the other.

Proof: As seen earlier, they are equal.

(c) \textbf{Under what circumstances are they equal?} They are never equal.

Reason: $\{\} \in 2^{(A)}$ but $\{\} \not\in (2^A)'$.

Proof: We know that $\{\}$ is a subset of every set, so in particular, it must be a subset of $A'$. Thus, by definition of power set, $\{\} \in 2^{(A)}$.

Additionally, $\{\}$ must be a subset of $A$, and thus $\{\} \in 2^A$. Therefore, by definition of complement, $\{\}$ cannot be a member of $(2^A)'$.

\textbf{Is one necessarily a subset of the other?} No, it is possible for neither to be a subset of the other.

Proof: Consider the case where $U = \{a, b\}$ and $A = \{a\}$. Then $(2^A)' = \{\} \cup \{a\} = \{\} \cup \{\{a\}, \{a, b\}\}$, and $2^{(A')} = 2^{\{b\}} = \{\} \cup \{b\}$. Each contains an element that is not contained in the other, so neither is a subset of the other.

3. Let $a$ and $b$ be even integers. By definition of even, we know that there exist integers $k$ and $m$ such that $a = 2k$ and $b = 2m$. Then $ab = 4km = 2(2km)$, which proves that $ab$ is even (with $2km$ as its witness).

4. Let $f : A \to B$ be a bijection. I will prove that $f^{-1}$ is a bijection and that $(f^{-1})^{-1} = f$. 

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To prove that $f^{-1}$ is a bijection, I must prove that it is injective and surjective.

- Proof of injectivity: Let $a_1, a_2 \in A$, and assume that $f^{-1}(a_1) = f^{-1}(a_2)$; I must show that $a_1 = a_2$.
  Let us define $z$ to be equal to $f^{-1}(a_1)$ and $f^{-1}(a_2)$. By the definition of inverse function, we see that $f(z) = a_1$ and $f(z) = a_2$. By transitivity of equality, we conclude that $a_1 = a_2$.

- Proof of surjectivity: Let $b \in B$; I must show that there exists an $a \in A$ such that $f(a) = b$.
  Let us define $a$ to be equal to $f^{-1}(b)$. By the definition of inverse function, we see that $f(a) = b$.

- Proof that $(f^{-1})^{-1} = f$: Let $a$ be an arbitrary member of $A$; I must show that $(f^{-1})^{-1}(a) = f(a)$.
  Let us define $c$ to be equal to $(f^{-1})^{-1}(a)$. By the definition of inverse function, we see that $f^{-1}(c) = a$. Now we can again use the definition of inverse function to see that $c = f(a)$. But $c = (f^{-1})^{-1}(a)$, so we can conclude that $(f^{-1})^{-1}(a) = f(a)$.

5. (Problem 1.45) The proof is by induction on $n$. We have two cases:

- Case where $n = 0$: We must prove that

$$
\sum_{i=1}^{0} \frac{1}{i(i+1)} = \frac{0}{0+1},
$$

which simplifies to $0 = 0$; this is obviously true.

- Case where $n = m + 1$ for some $m$: In this case, we may use the induction hypothesis

$$
\sum_{i=1}^{m} \frac{1}{i(i+1)} = \frac{m}{m+1},
$$

and we must prove that

$$
\sum_{i=1}^{m+1} \frac{1}{i(i+1)} = \frac{m+1}{m+2}.
$$
To do so, we simply rewrite the left hand side until it matches the right hand side:

$$\sum_{i=1}^{m+1} \frac{1}{i(i+1)}$$

$$= \frac{1}{(m+1)(m+2)} + \sum_{i=1}^{m} \frac{1}{i(i+1)} \quad \ldots \text{algebra} \ldots$$

$$= \frac{1}{(m+1)(m+2)} + \frac{m}{m+1} \quad \text{induction hypothesis}$$

$$= \frac{1 + m(m+2)}{(m+1)(m+2)} \quad \ldots \text{more algebra} \ldots$$

$$= \frac{(m+1)^2}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2}.$$ 

6. The proof is by contradiction; assume that there is a greatest odd integer. Call it $n$.

Now consider the integer $n+2$, which is obviously greater than $n$. Since $n$ is the greatest odd integer, $n + 2$ must not be odd. Next, we will reach a contradiction by proving that $n + 2$ is actually odd.

Since $n$ is odd, we know it is equal to $2k + 1$ for some integer $k$. Therefore, $n + 2 = 2k + 1 + 2 = 2(k + 1) + 1$, and this shows that $n + 2$ is odd (with $k + 1$ as its witness).