Theorem 0.1. **Weak Induction, Strong Induction and the Well Ordering Principle for Natural Numbers are equivalent.**

*Proof.* Let \( \{P(n)\}_{n \in \mathbb{N}} \) be a set of propositions, and let \( P(j \leq k) = P(1) \land \cdots \land P(k) \). Weak Induction is \( P(1) \land P(k) \to P(k+1) \forall k \) implies that \( P(k) \forall k \). Strong Induction is \( P(1) \land P(j \leq k) \to P(k+1) \forall k \), implies \( P(k) \forall k \). The Well Ordering Principle for Natural Numbers is every non-empty set of natural numbers contains a smallest element, i.e. \( \forall S \neq \emptyset \) such that \( S \subseteq \mathbb{N} \) there exists an \( i \in S \) such that \( i < j \forall j \in S \setminus \{i\} \). It suffices to show that a) Weak Induction implies Well Induction b) Weak Induction implies the Well Ordering Principle for Natural Numbers, and c) the Well Ordering Principle for Natural Numbers implies Strong Induction.

a) Strong Induction implies Weak Induction. Let \( P(n) \) be an arbitrary proposition such that \( P(1) \land P(k) \to P(k+1) \forall k \). We want to show that \( P(k) \forall k \). But \( P(k) \to P(k+1) \forall k \), so that in particular \( P(j \leq k) \to P(k+1) \forall k \). Hence, we have that \( P(1) \land P(j \leq k) \to P(k+1) \forall k \), so that by Strong Induction \( P(k) \forall k \).

b) Weak Induction implies the Well Ordering Principle for Natural Numbers. Towards a contradiction assume that Weak Induction is valid yet the Well Ordering Principle for Natural Numbers is false, i.e. \( P(1) \land P(k) \to P(k+1) \forall k \) implies that \( P(k) \forall k \). yet \( \exists S \neq \emptyset \subseteq \mathbb{N} \) such that \( \forall i \in S \exists j \in S \setminus \{i\} \) such that \( j \leq i \). Let \( R = \{i \in \mathbb{N} : i < j \forall j \in S \} \). Assume that \( 0 \notin \mathbb{N} \), so that the smallest element of \( \mathbb{N} \) is 1. Then \( S \neq \emptyset \) implies that \( R \neq \emptyset \), since otherwise 1 would also be the smallest element in \( S \). Moreover, \( R \cap S = \emptyset \) and by definition of \( R \) for every \( i \in R \) there does not exist a \( j \in S \) such that \( j < i \). So that \( i \in R \) implies \( i+1 \in R \), otherwise \( i + 1 \) would be the smallest element in \( S \). Let \( P(n) \) be the proposition that \( n \in R \). Then \( P(1) \land P(k) \to P(k+1) \forall k \). So that by Weak Induction \( P(k) \) for any \( k \), i.e. \( R = \mathbb{N} \). This contradicts the fact that \( \mathbb{N} \supseteq S \neq \emptyset \) and \( R \cap S = \emptyset \).

c) The Well Ordering Principle for Natural Numbers implies Strong Induction. Let \( P(n) \) be an arbitrary proposition such that \( P(1) \land P(k) \to P(k+1) \forall k \). Towards a contradiction assume that \( \exists S \neq \emptyset \subseteq \mathbb{N} \) such that \( \forall i \in S \exists j \in S \setminus \{i\} \) such that \( j \leq i \) yet \( \neg P(k) \) for some \( k \). Let \( S = \{n \leq k : \neg P(k)\} \). Then by the the Well Ordering Principle for Natural Numbers, there exists an \( i \in S \) such that \( i < j \forall j \in S \setminus i \). By the definition \( S \) and since \( i \in S \) we have that \( \neg P(i) \). But \( P(1) \) by assumption, so that \( i > 1 \). Hence \( i - 1 \geq 1 \). Moreover, \( j \notin i \forall j < i \), since \( i < j \forall j \in S \setminus \{i\} \), so that, \( P(j < i) \) by the definition of \( S \). But \( \neg P(i) \), hence \( P(j \leq i - 1) \neq P(i) \) contradicting the assumption that \( P(j \leq k) \to P(k+1) \forall k \). \( \square \)