**Definition.** For \( L \subseteq \{0,1\}^* \), define \( L/x = \{ y \in \{0,1\}^* | xy \in L \} \). \( xI_Ly \) if \( L/x = L/y \).

For \( x \in \{0,1\}^* \), let \( n_0(x) \) (respectively \( n_1(x) \)) be the number of 0’s (respectively 1’s) in the string \( x \).

5.13)
Let \( L = \{ x \in \{0,1\}^* | n_0(x) = n_1(x) \} \)
a & b) Claim: \( xI_Ly \) if and only if \( n_0(x) - n_1(x) = n_0(y) - n_1(y) \).
Proof: \( z \in L/x \) if and only if \( 0 = n_0(xz) - n_1(xz) = n_0(x) + n_0(z) - n_1(x) - n_1(z) \).
Hence, \( xI_Ly \) if and only if, for all \( z \in \{0,1\}^* \), \( n_0(x) + n_0(z) - n_1(x) - n_1(z) = n_0(y) + n_0(z) - n_1(y) - n_1(z) \) if and only if \( n_0(x) - n_1(x) = n_0(y) - n_1(y) \).

5.20)
Show that each of the following languages, \( L \), are not regular by showing that the relation \( I_L \) has an infinite number of equivalence classes. Hint: show that any two elements of the set \( \{0^n | n \geq 0 \} \) are distinguishable with respect to \( L \).
a) \( L = \{0^n10^{2n} | n \geq 0 \} \).
Proof: \( 10^{2n} \in L/0^n \) for any \( n \geq 0 \) implies that \( L/0^n = L/0^m \) only when \( n = m \).
c) \( L = \{0^i1^j | j = i, \text{ or } j = 2i \} \).
Proof: \( \{12^n, 1^n \} = \{ x \in L/0^n | n_0(x) = 0 \} \) for any \( n \geq 0 \) implies that \( L/0^n = L/0^m \) only when \( n = m \).
d) \( L = \{0^i1^j | j \text{ is a multiple of } i \} \).
Proof: For any \( n, k \geq 0 \) let \( L_k = \{1^{kn} \} \), so that \( \bigcup_{k=1}^{\infty} L_k = \{ x \in L/0^n | n_0(x) = 0 \} \). This implies that \( L/0^n = L/0^m \) only when \( n = m \).

5.23)
Use the Pumping Lemma to show that the following languages are not regular.
a) \( L = \{0^n10^{2n} | n \geq 0 \} \)
Hint: let \( x = 0^n10^{2n} \) and \( m = 2 \).
c) \( L = \{0^i1^j | j = i, \text{ or } j = 2i \} \).
Hint: let \( x = 0^n1^n \) and \( m = 2 \).
d) \( L = \{0^i1^j | j \text{ is a multiple of } i \} \).
Hint: let \( x = 0^n1^n \) and \( m = 2 \).

5.24)
a) Use the Pumping Lemma to show that \( L = \{ww | w \in \{0,1\}^* \} \) is not regular.
Hint: let \( x = 0^{n-1}10^{n-1} \) and \( m = 2 \).

5.26)
a) False: Let \( L_1 = \{0^n1^n | n \geq 0 \} \) and \( L_2 = \{0,1\}^* \). Then \( L_1 \subseteq L_2 \) and \( L_1 \) is not regular; however, \( L_2 \) is regular.
b) False: Let \( L_1 = \{A\} \) and \( L_2 = \{0^n1^n | n \geq 0 \} \). Then \( L_1 \subseteq L_2 \) and \( L_2 \) is not regular; however, \( L_1 \) is regular.
c) False: Let \( L_1 = \{x \in \{0,1\}^* | n_0(x) \geq n_1(x) \} \) and \( L_2 = \{x \in \{0,1\}^* | n_0(x) \leq n_1(x) \} \). Then neither \( L_1 \) nor \( L_2 \) is regular; however, \( L_1 \cup L_2 = \{0,1\}^* \) is regular.
i) False: For \( n = 1, 2, \ldots \) let \( L_n = \{0^n1^n \} \). Then all these languages are regular; however, \( \bigcup_{n=1}^{\infty} L_n = \{0^n1^n | n \geq 1 \} \) is not regular.