4.6) Claim: For any \( x \in \Sigma^* \), \( L = \{ \Lambda, x \} \) cannot be accepted by an NFA having only one accepting state.

First we note that any regular language can be accepted by an NFA - \( \Lambda \) with a single accepting state. Let \( \{Q, \Sigma, q_0, A, \delta \} \) be an NFA that accepts the language \( L \), let \( q' \) be a new state not in \( Q \), and define \( \delta' \) to be the same as \( \delta \) with the addition of \( \delta^*(q, \Lambda) = \{q'\} \) for every \( q \in A \). Then \( \{Q + q', \Sigma + \Lambda, q_0, \{q'\}, \delta'\} \) is an NFA - \( \Lambda \) with the single accepting state \( q' \) that accepts \( L \).

Proof: Toward a contradiction assume \( \{Q, \Sigma, q_0, \{q\}, \delta\} \) is an NFA accepting \( L \). \( \Lambda \in L \setminus \Sigma \), e.g. no \( \Lambda \) moves, implies that \( q = q_0 \) is the single accepting state. \( x \in L \) implies that \( q \in \delta^*(q_0, x) \). Combining these we get that 
\( q_0 \in \delta^*(q_0, x) \), which implies that \( xx \in L \). Contradiction.

4.7) Claim: Any regular language not containing \( \Lambda \) can be accepted by an NFA with a single accepting state.

Proof: Let \( \{Q, \Sigma, q_0, A, \delta\} \) be an NFA that accepts the language \( L \), let \( q' \) be a new state not in \( Q \), and define \( \delta' \) to be the same as \( \delta \) with the addition of \( \delta^*(q, a) = \{q'\} \) for every \( (q, a) \in Q \times \Sigma \) such that \( \delta^*(q, a) \cap A \neq \emptyset \). Note that since \( \Lambda \not\in L \) such an \( a \) exists for every \( x \in L \), i.e. \( a \) is the last element in \( x \). Then \( \{Q + q', \Sigma, q_0, \{q'\}, \delta'\} \) is an NFA with the single accepting state \( q' \) that accepts \( L \).

4.16) b) \( \Lambda(\{1\}) = \{1, 2, 5\}: 1 \xrightarrow{\Lambda} 2 \xrightarrow{\Lambda} 5 \)
d) \( \delta^*(1, ba) = \{3, 5\}: 1 \xrightarrow{\Lambda} 2 \xrightarrow{\Lambda} 5 \xrightarrow{b} 6 \xrightarrow{a} 5 \) and \( 1 \xrightarrow{\Lambda} 2 \xrightarrow{\Lambda} 5 \xrightarrow{b} 7 \xrightarrow{\Lambda} 1 \xrightarrow{\Lambda} 2 \xrightarrow{a} 3 \).

4.46) Let \( \Sigma_1 \) and \( \Sigma_2 \) be alphabets, and the function \( F : \Sigma_1^* \rightarrow \Sigma_2^* \) a homomorphism: i.e., \( f(xy) = f(x)f(y) \) for every \( x, y \in \Sigma_1^* \).

a) Claim: \( f(\Lambda) = \Lambda \).

Proof: Let \( a \in \Sigma_1^* \). Then \( f(a)\Lambda = f(a) = f(a\Lambda) = f(a)f(\Lambda) \). Hence, \( \Lambda = f(\Lambda) \).

b) For any language \( L \subseteq \Sigma_1^* \) define \( F(L) = \{ f(x) \mid x \in \Sigma_2^* \mid x \in L \} \).

Claim: \( F(L) \) is regular whenever \( L \subseteq \Sigma_1^* \) is regular.

Proof: By structural induction. By definition, \( |F(L)| = |L| \). For the base cases \( |F(L)| = |L| \leq 1 \). So that \( f(L) \) is finite and hence regular. Inductively assume that \( L_1, L_2 \subseteq \Sigma_1^* \) and \( f(L_1), f(L_2) \subseteq \Sigma_2^* \) are four regular languages. It suffices to show that (i) \( f(L_1L_2) \) is regular, (ii) \( f(L_1 \cup L_2) \) is regular, and (iii) \( f(L_1^*) \) is regular.

i) \( f(L_1L_2) = \{ f(x_1)f(x_2) \mid x_1 \in L_1, x_2 \in L_2 \} = f(L_1)f(L_2) \) which is regular since concatenation of two regular languages, e.g. \( f(L_1), f(L_2), \) is a regular language.

ii) \( f(L_1 \cup L_2) = \{ f(x) \mid x \in L_1 \cup L_2 \} = \{ f(x) \mid x \in L_1 \} \cup \{ f(x) \mid x \in L_2 \} = f(L_1) \cup f(L_2) \) which is regular since the union of two regular languages, e.g. \( f(L_1), f(L_2), \) is a regular language.

iii) \( f(L_1^*) = \{ f(x) \mid x \in L_1^* \} = \bigcup_{i=0}^{\infty} \{ f(x) \mid x \in L_1^i \} = \bigcup_{i=0}^{\infty} f(L_1^i) = f(L_1)^* \) which is regular since the Kleene * of a regular language, e.g. \( f(L_1), \) is a regular language.

c) For any language \( L \subseteq \Sigma_2^* \) define \( f^{-1}(L) = \{ x \in \Sigma_1^* \mid f(x) \in L \}, \) i.e., for all \( x \in \Sigma_2^* \) \( f^{-1}(x) = \{ y \in \Sigma_1^* \mid f(y) = x \} \) and \( f^{-1}(L) = \{ f^{-1}(x) \mid x \in L \}. \)

Claim: \( f^{-1}(L) \) is regular whenever \( L \subseteq \Sigma_2^* \) is regular.
Proof: Let \( \{Q_2, \Sigma_2, q_0, A, \delta_2\} \) be a FA accepting \( L \), and define \( \delta_1 : Q \times \Sigma_1 \to Q \) such that for all \( q \in Q \), for all \( a \in \Sigma_2 \), and for all \( x \in f^{-1}(a) \), \( \delta_1(q, x) = \delta_2(q, a) \). Moreover, initialize \( Q_1 = Q_2 \) and for all \( q \in Q \), for all \( a \in \Sigma_1 \), and for all \( x \in \Sigma_1^* \) such that \( \delta_1(q, ax) \in Q_1 \), but \( \delta_1(q, a) \notin Q_1 \), add a new state \( q' \) to \( Q_1 \) and let \( \delta_1(q', a) = \{q'\} \) and \( \delta_1(q', x) = \delta_1(q, ax) \). Then \( \{Q, \Sigma_1, q_0, A, \delta_1\} \) is an NFA which accepts the language \( f^{-1}(L) \).

Note that \( x \in L \) if and only if \( \delta_2(q_0, x) \in A \). In one direction, \( \delta_2(q_0, x) \in A \) only if \( \delta_1(q_0, y) \in A \) for every \( y \in f^{-1}(x) \) by construction. In the other direction, if \( \delta_1(q_0y) \in A \) then there exist \( x_i \in \Sigma_2^*, y_i \in f^{-1}(x_i) \), and \( q_i \in Q_2 \) for \( 1 \leq i \leq n \) such that \( y = y_1, \ldots, y_n, q_n \in A \), and \( \Sigma_2(q_{i-1}, x_i) = q_i \) for \( 1 \leq i \leq n \). But this implies that, setting \( x = x_1 \cdots x_n \in \Sigma_2^* \), \( \delta_2(q_0, x) = q_n \in A \). Moreover, since \( f^{-1}(xy) = \{wz \in \Sigma_1^* | f(wz) = f(w)f(z) = xy\} \supseteq \{wz \in \Sigma_1^* | f(w) = x, f(z) = y\} = \{w \in \Sigma_1^* | f(w) = x\} \{z \in \Sigma_1^* | f(z) = y\} = f^{-1}(x)f^{-1}(y) \), it follows that \( y \in f^{-1}(x) \). So that, to paraphrase, if \( \delta_1(q_0y) \in A \) then there exist an \( x \in \Sigma_2^* \) such that \( \delta_2(q_0, x) \in A \) and \( y \in f^{-1}(x) \).