Here we show how to convert an $\epsilon$-NFA into a (normal) NFA. First, recall the definitions:

- An NFA is a tuple $(Q, \Sigma, \delta, q_0, F)$ where $Q$ is a set of states, $\Sigma$ is the alphabet, $\delta$ is the transition function that maps each state-symbol pair to a subset of $Q$, $q_0$ is the initial state, and $F \subset Q$ is the set of final (or accepting) states.

- We extend the transition function $\delta$ for an NFA to apply to strings as well as symbols:
  
  $\delta(q, \epsilon) = q$
  
  if $|w| \geq 1$ then $w = xa$ for some string $x$ and symbol $a$, and
  
  $\delta(q, w) = \delta(q, xa) = \bigcup_{r \in \delta(q, x)} \delta(r, a)$

- A string $w$ is accepted by an NFA if $\delta(q_0, w) \cap F \neq \emptyset$, i.e. $\delta(q_0, w)$ contains a state that is also in $F$.

- An $\epsilon$-NFA is a tuple $(Q, \Sigma, \delta, q_0, F)$ where $Q$ is a set of states, $\Sigma$ is the alphabet, $\delta$ is the transition function that maps each pair consisting of a state and a symbol in $\Sigma \cup \{\epsilon\}$ to a subset of $Q$, $q_0$ is the initial state, and $F \subset Q$ is the set of final (or accepting) states.

- The $\epsilon$-closure of a state $q$ is $\text{eclose}(q)$ and contains all states reachable from $q$ by following (zero or more) transitions labeled by $\epsilon$. Thus $q$ is always in $\text{eclose}(q)$.

- The $\epsilon$-closure of a set of states $S \subseteq Q$ is defined by:
  
  $\text{eclose}(S) = \bigcup_{r \in S} \text{eclose}(r)$

- Fact 1: if $S$ and $T$ are subsets of states then $\text{eclose}(S \cup T) = \text{eclose}(S) \cup \text{eclose}(T)$.

- We extend the transition function $\delta$ for an $\epsilon$-NFA to apply to strings in $\Sigma^*$. Note that (unlike for DFA’s and normal NFA’s) the extension for $\epsilon$-NFAs can be different from $\delta$ on single symbols (including $\epsilon$), so we must use a different symbol for it, we use $\hat{\delta}$.

  $\hat{\delta}(q, \epsilon) = \text{eclose}(q)$

  if $|w| \geq 1$ then $w = xa$ for some string $x$ and symbol $a$, and

  $\hat{\delta}(q, w) = \hat{\delta}(q, xa) = \text{eclose}\left(\bigcup_{r \in \hat{\delta}(q, x)} \delta(r, a)\right)$
• A string $w$ is accepted by an $\epsilon$-NFA if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.

Our goal is to show the following theorem:

**Theorem 1.** The set of languages accepted by NFAs is the same as the set of languages accepted by $\epsilon$-NFAs.

The proof is in two parts. First we show the easy part that every language accepted by a NFA is also accepted by an $\epsilon$-NFA. This is because every NFA is an $\epsilon$-NFA with no $\epsilon$-transitions.

**Claim 1.** If $N = (Q, \Sigma, q_0, F)$ is an NFA, then $E = (Q, \Sigma, \delta_E, q_0, F)$ where $\delta_E(q, a) = \delta(q, a)$ for all $a \in \Sigma$ and $\delta_E(q, \epsilon) = \emptyset$ for all $q \in Q$ is an $\epsilon$-NFA accepting the same language.

**Proof.** Since $E$ contains no $\epsilon$-transitions, $\text{eclose}(S) = S$ for any set of states $S \subseteq Q$. Therefore, $\delta_E$ is defined by:

- $\hat{\delta}_E(q, \epsilon) = \text{eclose}(q) = q$
- if $|w| \geq 1$ then $w = xa$ for some string $x$ and symbol $a$, and

$\hat{\delta}_E(q, w) = \hat{\delta}_E(q, xa) = \text{eclose} \left( \bigcup_{r \in \delta(q, x)} \delta_E(r, a) \right) = \bigcup_{r \in \delta_E(q, x)} \delta_E(r, a) = \bigcup_{r \in \delta_E(q, x)} \delta(r, a)$

and $\hat{\delta}_E$ matches the definition of the extended $\delta$ in the NFA $N$, so they are the same. Therefore, for any word $w \in \Sigma^*$, $\hat{\delta}_E(q_0, w) \cap F \neq \emptyset$ for exactly those $w$ where $\delta(q_0, w) \cap F \neq \emptyset$, and both $N$ and $E$ accept the same language. \qed

**Claim 2.** Given any $\epsilon$-NFA $E = (Q, \Sigma, \delta_E, q_0, F)$ one can construct an NFA $N = (Q, \Sigma, \delta_N, q_0, F_N)$ accepting the same language. The construction sets

$\delta_N(q, a) = \bigcup_{r \in \text{eclose}(q)} \delta_E(r, a)$

and $F_N$ to the set of states $q$ such that $\text{eclose}(q) \cap F \neq \emptyset$.

Note that $N$ and $E$ share state sets, starting states, and alphabets. Also, all closure operations are with respect to $E$ and $\delta_E$ (recall that NFA $N$ has no $\epsilon$-transitions).

**Proof.** First notice that $N$ accepts a string $w$ if (and only if) $\delta_N(q_0, w)$ contains a state in $F_N$. By the definition of $F_N$, this is equivalent to $\text{eclose}(\delta_N(q_0, w))$ contains a state in $F$. Also, $E$ accepts a string $w$ if (and only if) $\delta_E(q_0, w)$ contains a state in $F$. Therefore if $\text{eclose}(\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w)$ for all $s \in \Sigma^*$ then $E$ and $N$ accept the same language.

We now prove that for any string $w$ that $\text{eclose}(\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w)$ by induction on $|w|$.

For each $n \geq 0$, define IH($n$) as: $\text{eclose}(\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w)$ for all $w \in \Sigma^n$. 

2
Base Case: show IH(0).
There is only one string in $\Sigma^0$, the empty string $\epsilon$.

\[
\text{ECLOSE} (\delta_N(q_0, \epsilon)) = \text{ECLOSE} (q_0) \quad \text{since } \delta_N(q, \epsilon) = q \\
= \hat{\delta}_E(q_0, \epsilon) \quad \text{def. } \hat{\delta}_E(q, \epsilon)
\]

Inductive Step: Assume $n \geq 0$ and IH($n$) to show IH($n+1$). Let $w$ be an arbitrary string in $\Sigma^{n+1}$. Then $w$ can be written as $xa$ for some string $x \in \Sigma^n$ and symbol $a$. We now examine ECLOSE($\delta_N(q_0, w)$).

\[
\begin{align*}
\text{ECLOSE} (\delta_N(q_0, w)) & = \text{ECLOSE} (\delta_N(q_0, xa)) \quad \text{since } w = xa \\
& = \text{ECLOSE} \left( \bigcup_{r \in \text{ECLOSE} (\delta_N(q_0, x))} \delta_E(r, a) \right) \quad \text{def. } \delta_N \\
& = \text{ECLOSE} \left( \bigcup_{r \in \delta_E(q_0, x)} \delta_E(r, a) \right) \quad \text{using IH($n$) on } x \\
& = \bigcup_{r \in \delta_E(q_0, x)} \text{ECLOSE} (\delta_E(r, a)) \quad \text{Fact 1} \\
& = \hat{\delta}_E(q_0, xa) \quad \text{def. } \hat{\delta}_E \\
& = \hat{\delta}_E(q_0, w) \quad \text{since } w = xa
\end{align*}
\]

Thus $\text{ECLOSE} (\delta_N(q_0, w)) = \hat{\delta}_E(q_0, w)$, showing IH($n + 1$) and finishing the proof.  

The proof of Theorem 1 follows immediately from Claim 1 and Claim 2.