Claim. GS matching $S^*$ is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.
- Let $Y$ be first such man, and let $A$ be first valid woman that rejects him.
- Let $S$ be a stable matching where $A$ and $Y$ are matched.
- When $Y$ is rejected, $A$ forms (or reaffirms) engagement with a man, say $Z$, whom she prefers to $Y$.
- Let $B$ be $Z$'s partner in $S$.
- $Z$ not rejected by any valid partner at the point when $Y$ is rejected by $A$. Thus, $Z$ prefers $A$ to $B$.
- But $A$ prefers $Z$ to $Y$.
- Thus $A-Z$ is unstable in $S$. ■
Stable Matching Summary

**Stable matching problem.** Given preference profiles of n men and n women, find a stable matching. 

no man and woman prefer to be with each other than assigned partner

**Gale-Shapley algorithm.** Finds a stable matching in $O(n^2)$ time.

**Man-optimality.** In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

**Q.** Does man-optimality come at the expense of the women?
Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching $S^\ast$.

Pf.
- Suppose A-Z matched in $S^\ast$, but Z is not worst valid partner for A.
- There exists stable matching $S$ in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in $S$.
- Z prefers A to B.
- Thus, A-Z is an unstable in $S$. □
Ex: Men $\sim$ hospitals, Women $\sim$ med school residents.

Variant 1. Some participants declare others as unacceptable.

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.
Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)
- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.
- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem (1986). Any hospital that has some empty positions at some stable matching is assigned precisely the same set of residents at every stable matching.
Q. Can there be an incentive to misrepresent your preference profile?
   - Assume you know men’s propose-and-reject algorithm will be run.
   - Assume that you know the preference profiles of all other participants.

**Fact.** No, for any man yes, for some women. No mechanism can guarantee a stable matching and be cheatproof.

### Men’s Preference List

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<tr>
<td>Xavier</td>
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<td>B</td>
<td>C</td>
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<td>Yancey</td>
<td>B</td>
<td>A</td>
<td>C</td>
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<tr>
<td>Zeus</td>
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### Women’s True Preference Profile

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<td>Bertha</td>
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### Amy Lies

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Lessons Learned

Powerful ideas learned in course.
- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]
1.2 Five Representative Problems
Interval Scheduling

**Input.** Set of jobs with start times and finish times.

**Goal.** Find **maximum cardinality** subset of mutually compatible jobs.
Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.
Bipartite Matching

Input. Bipartite graph.
Goal. Find maximum cardinality matching.
Independent Set

**Input.** Graph.

**Goal.** Find **maximum cardinality** independent set.

subset of nodes such that no two joined by an edge
Competitive Facility Location

**Input.** Graph with weight on each each node.

**Game.** Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

**Goal.** Select a *maximum weight* subset of nodes.

Second player can guarantee 20, but not 25.