Dynamic Programming Summary

Recipe.
- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.
- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares.
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Top-down vs. bottom-up: different people have different intuitions.
6.6 Sequence Alignment
String Similarity

How similar are two strings?

- occurrence
- occurrence

6 mismatches, 1 gap

1 mismatch, 1 gap

0 mismatches, 3 gaps
Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.


- Gap penalty $\delta$; mismatch penalty $\alpha_{pq}$.
- Cost = sum of gap and mismatch penalties.

$$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$$

$$2\delta + \alpha_{CA}$$
Sequence Alignment

Goal: Given two strings $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_n$ find alignment of minimum cost.

Def. An alignment $M$ is a set of ordered pairs $x_i$-$y_j$ such that each item occurs in at most one pair and no crossings.

Def. The pair $x_i$-$y_j$ and $x_{i'}$-$y_{j'}$ cross if $i < i'$, but $j > j'$.

Ex: CTACCG vs. TACATG.
Sol: $M = x_2$-$y_1$, $x_3$-$y_2$, $x_4$-$y_3$, $x_5$-$y_4$, $x_6$-$y_6$.

$$\text{cost}(M) = \sum_{(x_i, y_j) \in M} \alpha_{x_i, y_j} + \sum_{i : x_i \text{ unmatched}} \delta_i + \sum_{j : y_j \text{ unmatched}} \delta_j$$

\begin{align*}
\text{cost}(M) &= \sum_{(x_i, y_j) \in M} \alpha_{x_i, y_j} + \sum_{i : x_i \text{ unmatched}} \delta_i + \sum_{j : y_j \text{ unmatched}} \delta_j \\
&= \alpha_{x_2, y_1} + \delta_{x_1} + \alpha_{x_3, y_2} + \delta_{y_1} + \alpha_{x_4, y_3} + \delta_{y_2} + \alpha_{x_5, y_4} + \delta_{y_3} + \alpha_{x_6, y_5} + \delta_{y_4} \\
&= \alpha_{x_2, y_1} + \delta_{x_1} + \alpha_{x_3, y_2} + \delta_{y_1} + \alpha_{x_4, y_3} + \delta_{y_2} + \alpha_{x_5, y_4} + \delta_{y_3} + \alpha_{x_6, y_5} + \delta_{y_4} \\
&= \alpha_{x_2, y_1} + \alpha_{x_3, y_2} + \alpha_{x_4, y_3} + \alpha_{x_5, y_4} + \alpha_{x_6, y_5} + \delta_{x_1} + \delta_{y_1} + \delta_{y_2} + \delta_{y_3} + \delta_{y_4}
\end{align*}
Sequence Alignment: Problem Structure

**Def.** $OPT(i, j) = \text{min cost of aligning strings } x_1 x_2 \ldots x_i \text{ and } y_1 y_2 \ldots y_j$.

- **Case 1:** $OPT$ matches $x_i$-$y_j$.
  - pay mismatch for $x_i$-$y_j$ + min cost of aligning two strings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$

- **Case 2a:** $OPT$ leaves $x_i$ unmatched.
  - pay gap for $x_i$ and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$

- **Case 2b:** $OPT$ leaves $y_j$ unmatched.
  - pay gap for $y_j$ and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$

\[
OPT(i, j) = \begin{cases} 
  j\delta & \text{if } i = 0 \\
  \min \left\{ \alpha_{x_i,y_j} + OPT(i-1, j-1), \delta + OPT(i-1, j), \delta + OPT(i, j-1) \right\} & \text{otherwise} \\
  i\delta & \text{if } j = 0 
\end{cases}
\]
Sequence Alignment: Algorithm

Sequence-Alignment(m, n, x₁x₂...xₘ, y₁y₂...yₙ, δ, α)
{
    for i = 0 to m
        M[0, i] = iδ
    for j = 0 to n
        M[j, 0] = jδ

    for i = 1 to m
        for j = 1 to n
            M[i, j] = min(α[xᵢ, yⱼ] + M[i-1, j-1],
                            δ + M[i-1, j],
                            δ + M[i, j-1])

    return M[m, n]
}

Analysis. Θ(mn) time and space.

English words or sentences: m, n ≤ 10.

Computational biology: m = n = 100,000. 10 billions ops OK, but 10GB array?
6.7 Sequence Alignment in Linear Space
Q. Can we avoid using quadratic space?

Easy. Optimal value in $O(m + n)$ space and $O(mn)$ time.
- Compute $OPT(i, \cdot)$ from $OPT(i-1, \cdot)$.
- No longer a simple way to recover alignment itself.

Theorem. [Hirschberg 1975] Optimal alignment in $O(m + n)$ space and $O(mn)$ time.
- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.
Sequence Alignment: Linear Space

- Edit distance graph.
  - Let $f(i, j)$ be shortest path from $(0,0)$ to $(i, j)$.
  - Observation: $f(i, j) = \text{OPT}(i, j)$.

![Edit distance graph with nodes and edges labeled with symbols and distances.]
Edit distance graph.

- Let \( f(i, j) \) be shortest path from \((0,0)\) to \((i, j)\).
- Can compute \( f(\cdot, j) \) for any \( j \) in \( O(mn) \) time and \( O(m + n) \) space.
Sequence Alignment: Linear Space

Edit distance graph.

- Let $g(i, j)$ be shortest path from $(i, j)$ to $(m, n)$.
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and $(m, n)$
Sequence Alignment: Linear Space

Edit distance graph.

- Let $g(i, j)$ be shortest path from $(i, j)$ to $(m, n)$.
- Can compute $g(\bullet, j)$ for any $j$ in $O(mn)$ time and $O(m + n)$ space.
Observation 1. The cost of the shortest path that uses \((i, j)\) is \(f(i, j) + g(i, j)\).
Observation 2. Let $q$ be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, the shortest path from $(0, 0)$ to $(m, n)$ uses $(q, n/2)$. 
Sequence Alignment: Linear Space

**Divide:** find index $q$ that minimizes $f(q, n/2) + g(q, n/2)$ using DP.
- Align $x_q$ and $y_{n/2}$.

**Conquer:** recursively compute optimal alignment in each piece.
Sequence Alignment: Running Time Analysis Warmup

**Theorem.** Let $T(m, n) = \max$ running time of algorithm on strings of length at most $m$ and $n$. $T(m, n) = O(mn \log n)$.

**Remark.** Analysis is not tight because two sub-problems are of size $(q, n/2)$ and $(m - q, n/2)$. In next slide, we save log $n$ factor.
Sequence Alignment: Running Time Analysis

Theorem. Let $T(m, n) = \max$ running time of algorithm on strings of length $m$ and $n$. $T(m, n) = O(mn)$.

Pf. (by induction on $n$)

- $O(mn)$ time to compute $f(\cdot, n/2)$ and $g(\cdot, n/2)$ and find index $q$.
- $T(q, n/2) + T(m - q, n/2)$ time for two recursive calls.
- Choose constant $c$ so that:

- Base cases: $m = 2$ or $n = 2$.
- Inductive hypothesis: $T(m, n) \leq 2cmn$.

\[
\begin{align*}
T(m, 2) & \leq cm \\
T(2, n) & \leq cn \\
T(m, n) & \leq cmn + T(q, n/2) + T(m - q, n/2)
\end{align*}
\]

\[
\begin{align*}
T(m,n) & \leq T(q,n/2) + T(m - q, n/2) + cmn \\
& \leq 2cqn/2 + 2c(m - q)n/2 + cmn \\
& = cqn + cmn - cqn + cmn \\
& = 2cmn
\end{align*}
\]
6.8 Shortest Paths
**Shortest Paths**

**Shortest path problem.** Given a directed graph $G = (V, E)$, with edge weights $c_{vw}$, find shortest path from node $s$ to node $t$. 

![Diagram of a graph with nodes and edges labeled with weights.](image)

**Ex.** Nodes represent agents in a financial setting and $c_{vw}$ is cost of transaction in which we buy from agent $v$ and sell immediately to $w$. 

allow negative weights
Shortest Paths: Failed Attempts

**Dijkstra.** Can fail if negative edge costs.

**Re-weighting.** Adding a constant to every edge weight can fail.
Shortest Paths: Negative Cost Cycles

Negative cost cycle.

Observation. If some path from $s$ to $t$ contains a negative cost cycle, there does not exist a shortest $s$-$t$ path; otherwise, there exists one that is simple.
Shortest Paths: Dynamic Programming

Def. \( \text{OPT}(i, v) = \text{length of shortest } v-t \text{ path } P \text{ using at most } i \text{ edges.} \)

- Case 1: \( P \) uses at most \( i-1 \) edges.
  - \( \text{OPT}(i, v) = \text{OPT}(i-1, v) \)

- Case 2: \( P \) uses exactly \( i \) edges.
  - if \( (v, w) \) is first edge, then \( \text{OPT} \) uses \( (v, w) \), and then selects best \( w-t \) path using at most \( i-1 \) edges

\[
\text{OPT}(i, v) = \begin{cases} 
\infty & \text{if } i = 0 \\
\min \left\{ \text{OPT}(i-1, v), \min_{(v, w) \in E} \{ \text{OPT}(i-1, w) + c_{vw} \} \right\} & \text{otherwise}
\end{cases}
\]

Remark. By previous observation, if no negative cycles, then \( \text{OPT}(n-1, v) = \text{length of shortest } v-t \text{ path.} \)
Shortest-Path(G, t) {
    foreach node v ∈ V
        M[0, v] ← ∞
    M[0, t] ← 0

    for i = 1 to n-1
        foreach node v ∈ V
            M[i, v] ← M[i-1, v]
        foreach edge (v, w) ∈ E
            M[i, v] ← min { M[i, v], M[i-1, w] + c_{vw} }
}

Analysis. \( \Theta(mn) \) time, \( \Theta(n^2) \) space.

Finding the shortest paths. Maintain a "successor" for each table entry.
Practical improvements.

- Maintain only one array $M[v] =$ shortest $v$-$t$ path that we have found so far.
- No need to check edges of the form $(v, w)$ unless $M[w]$ changed in previous iteration.

**Theorem.** Throughout the algorithm, $M[v]$ is length of some $v$-$t$ path, and after $i$ rounds of updates, the value $M[v]$ is no larger than the length of shortest $v$-$t$ path using $\leq i$ edges.

**Overall impact.**

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.
Bellman-Ford: Efficient Implementation

Push-Based-Shortest-Path(G, s, t) {
    foreach node v ∈ V {
        M[v] ← ∞
        successor[v] ← φ
    }
    M[t] = 0
    for i = 1 to n-1 {
        foreach node w ∈ V {
            if (M[w] has been updated in previous iteration) {
                foreach node v such that (v, w) ∈ E {
                    if (M[v] > M[w] + c_{vw}) {
                        M[v] ← M[w] + c_{vw}
                        successor[v] ← w
                    }
                }
            }
        }
        If no M[w] value changed in iteration i, stop.
    }
}
6.9 Distance Vector Protocol
Distance Vector Protocol

Communication network.
- Nodes ≈ routers.
- Edges ≈ direct communication link.
- Cost of edge ≈ delay on link. ← naturally nonnegative, but Bellman-Ford used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman-Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each foreach loop executes is not important. Moreover, algorithm still converges even if updates are asynchronous.
Distance Vector Protocol

Distance vector protocol.
- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs $n$ separate computations, one for each potential destination node.
- "Routing by rumor."

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge costs may change during algorithm (or fail completely).
Path Vector Protocols

Link state routing.
- Each router also stores the entire path.
- Based on Dijkstra’s algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).
6.10 Negative Cycles in a Graph
Detecting Negative Cycles

Lemma. If $OPT(n,v) = OPT(n-1,v)$ for all $v$, then no negative cycles.

Lemma. If $OPT(n,v) < OPT(n-1,v)$ for some node $v$, then (any) shortest path from $v$ to $t$ contains a cycle $W$. Moreover $W$ has negative cost.

Pf. (by contradiction)
- Since $OPT(n,v) < OPT(n-1,v)$, we know $P$ has exactly $n$ edges.
- By pigeonhole principle, $P$ must contain a directed cycle $W$.
- Deleting $W$ yields a $v$-$t$ path with $< n$ edges $\Rightarrow$ $W$ has negative cost.
Detecting Negative Cycles

**Theorem.** Can detect negative cost cycle in $O(mn)$ time.

- Add new node $t$ and connect all nodes to $t$ with 0-cost edge.
- Check if $OPT(n, v) = OPT(n-1, v)$ for all nodes $v$.
  - if yes, then no negative cycles
  - if no, then extract cycle from shortest path from $v$ to $t$
Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!
Detecting Negative Cycles: Summary

Bellman-Ford. $O(mn)$ time, $O(m + n)$ space.

- Run Bellman-Ford for $n$ iterations (instead of $n-1$).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 288 for improved version and early termination rule.