Due Tuesday 1/22/02

Prove each of the following statements by induction.

1. (10 Points) \[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \] for all \( n \geq 1 \).

2. (10 Points) \( n^5 - n \) is divisible by 5 for all \( n \geq 0 \).

3. (20 Points) Define \( A_n \) by the recurrence

\[
\begin{aligned}
A_0 &= 0 \\
A_1 &= 1 \\
A_n &= A_{n-1} + 2A_{n-2} \quad \text{for } n \geq 2
\end{aligned}
\]

Show that \( A_n = \frac{1}{3} \left[ 2^n - (-1)^n \right] \) for all \( n \geq 0 \). (Hint: use double induction.)

4. (20 Points) Prove the **Binomial Theorem**: Given real numbers \( x \) and \( y \), then for all \( n \geq 0 \):

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) are the so called combinatorial coefficients. Hint: use Pascal’s identity:

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.
\]

5. (20 Points) A graph is called **complete** if every pair of distinct vertices are joined by exactly one edge. Prove that for all \( n \geq 1 \), if \( G \) is a complete graph on \( n \) vertices, then \( G \) has exactly \( n(n-1)/2 \) edges.