Figure 9.1 Analysis of the algorithm SELECT. The \( n \) elements are represented by small circles, and each group of 5 elements occupies a column. The medians of the groups are whitened, and the median-of-medians \( x \) is labeled. (When finding the median of an even number of elements, we use the lower median.) Arrows go from larger elements to smaller, from which we can see that 3 out of every full group of 5 elements to the right of \( x \) are greater than \( x \), and 3 out of every group of 5 elements to the left of \( x \) are less than \( x \). The elements known to be greater than \( x \) appear on a shaded background.

step 2 are greater than or equal to the median-of-medians \( x \). Thus, at least half of the \( \lceil n/5 \rceil \) groups contribute at least 3 elements that are greater than \( x \), except for the one group that has fewer than 5 elements if 5 does not divide \( n \) exactly, and the one group containing \( x \) itself. Discounting these two groups, it follows that the number of elements greater than \( x \) is at least

\[
3 \left( \left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2 \right) \geq \frac{3n}{10} - 6.
\]

Similarly, at least \( 3n/10 - 6 \) elements are less than \( x \). Thus, in the worst case, step 5 calls SELECT recursively on at most \( 7n/10 + 6 \) elements.

We can now develop a recurrence for the worst-case running time \( T(n) \) of the algorithm SELECT. Steps 1, 2, and 4 take \( O(n) \) time. (Step 2 consists of \( O(n) \) calls of insertion sort on sets of size \( O(1) \).) Step 3 takes time \( T(\lceil n/5 \rceil) \), and step 5 takes time at most \( T(7n/10 + 6) \), assuming that \( T \) is monotonically increasing. We make the assumption, which seems unmotivated at first, that any input of fewer than 140 elements requires \( O(1) \) time; the origin of the magic constant 140 will be clear shortly. We can therefore obtain the recurrence

---

1Because of our assumption that the numbers are distinct, all medians except \( x \) are either greater than or less than \( x \).